A Graph Theoretic Approach to Regularity of Toric Ideals

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Motivation

One measure of the complexity of an ideal is its *regularity*, which may be difficult to compute directly. To study the regularity of *toric ideals* (prime ideals generated by differences of monomials), we make use of correspondences between graphs and ideals. In particular, we determine when and how we can apply Theorem 1 to compute the regularity of toric ideals generated by chordal bipartite graphs.

Connecting Ideals and Graphs

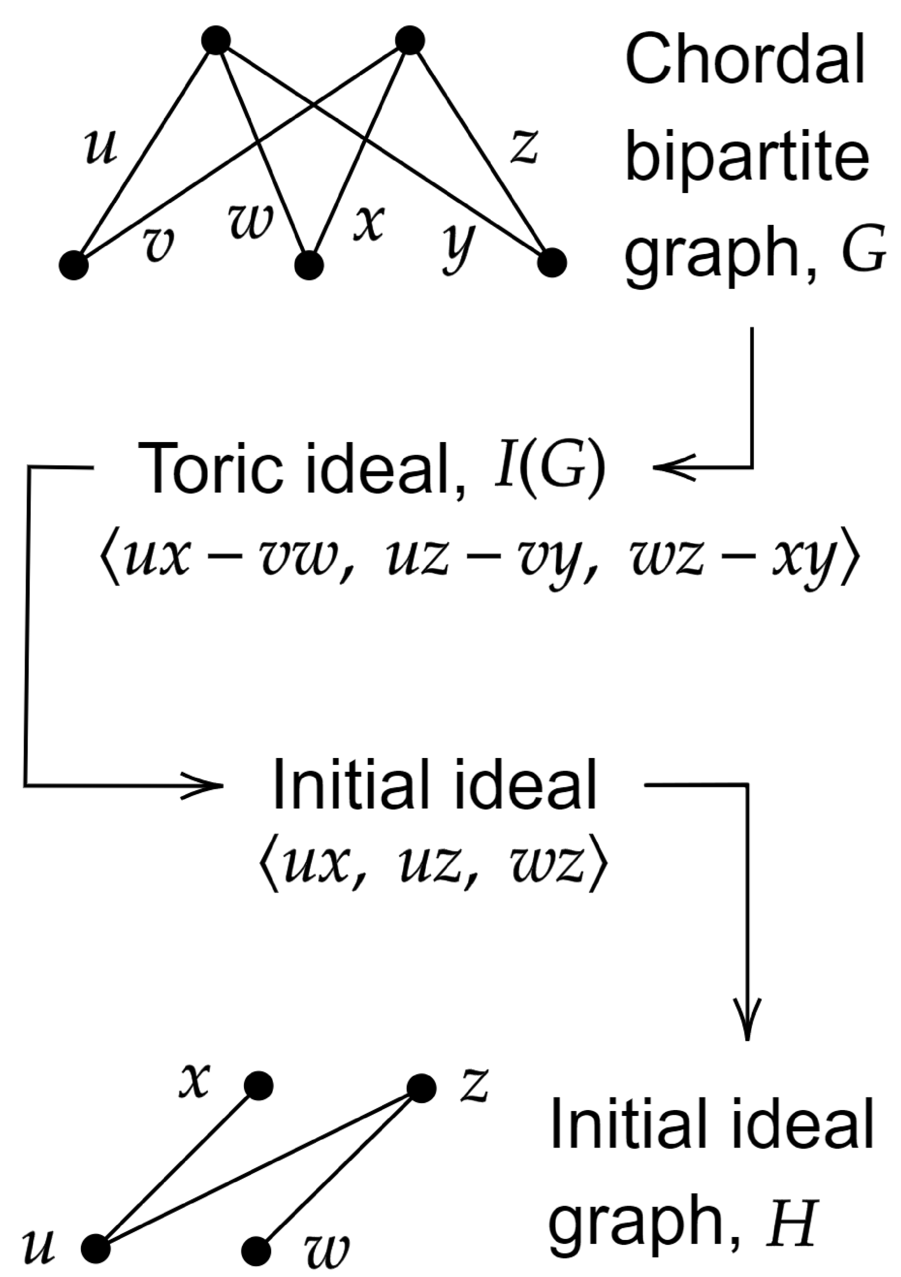

Fig. 1: A diagram illustrating how a toric ideal, generated from a chordal bipartite graph G, is associated to a new graph, H.

A Guiding Result

Let $I(G)$ denote the toric ideal generating the initial ideal graph H and let $\mu(H)$ be the induced matching number of H.

Theorem 1. *[1], [2] If H is pure vertex decomposable and induced 5-cycle free, then* $\operatorname{reg}(I(G)) = \mu(H) + 1$.

In order to apply this theorem, we classify initial ideal graphs and investigate their properties.

Down-Left Graphs

Definition 1. *A **down-left graph**, denoted $G(n, m)$, is a graph that can be arranged as an $n \times m$ grid of vertices where each vertex is adjacent to every other vertex that is down and to the left of it.*

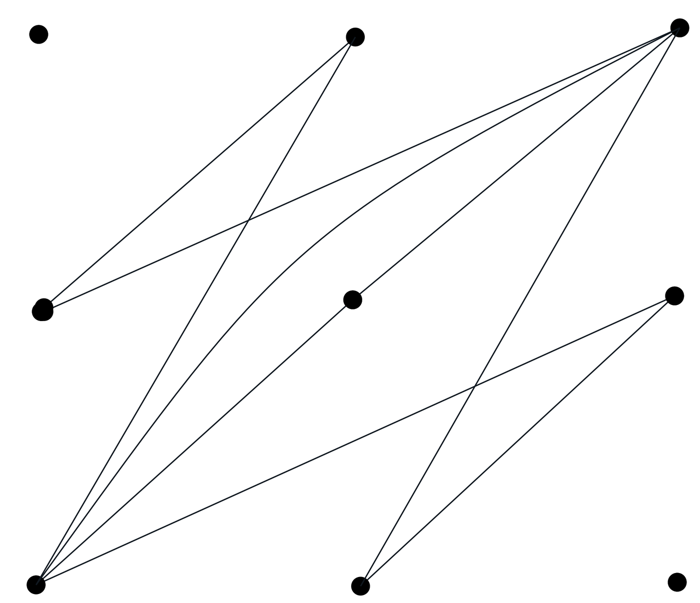

Fig. 2: $G(3, 3)$.

Theorem 2. $G(n, m)$ *is pure vertex decomposable for all n, m.*

This is because at each step in any vertex decomposition, the graph is pure. A maximal independent set in $G(n, m)$ is a "down-right path" as illustrated in Figure 3. Furthermore, in every down-left graph, whenever there is a 5-cycle, there must also be a chord, as illustrated in Figure 4. Thus, $G(n, m)$ is induced 5-cycle free. [3]

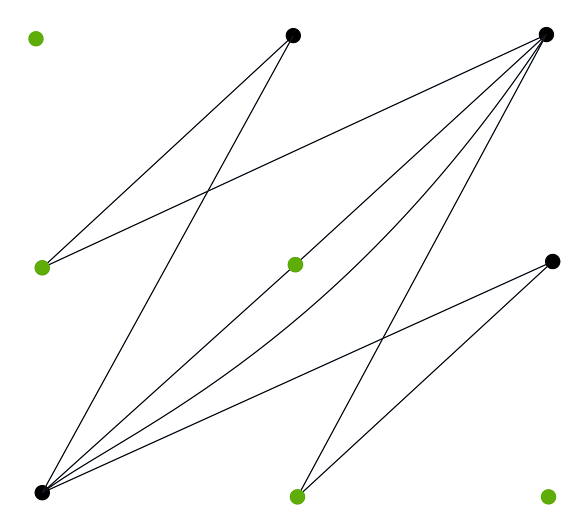

Fig. 3: $G(3, 3)$ with a maximal independent set in green.

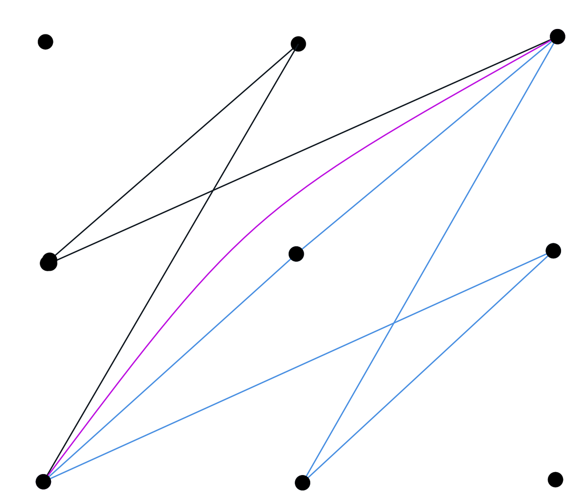

Fig. 4: $G(3, 3)$ with a blue 5-cycle and fuchsia chord.

References

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Classifying Initial Ideal Graphs

Theorem 3. *A chordal bipartite graph G contains an induced "conga drum graph" ($K_{3,3}$ minus any edge) if and only if G generates an initial ideal graph H that contains an induced 5-cycle.*

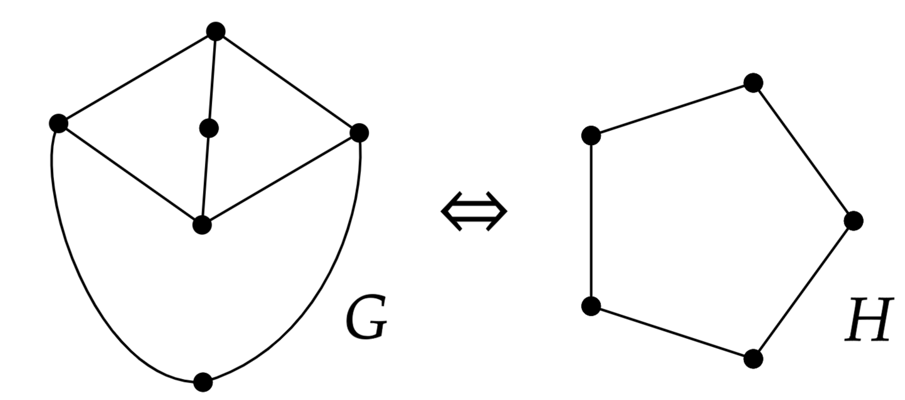

Fig. 5: A depiction of Theorem 3.

Theorem 4. *If a chordal bipartite graph G does not contain an induced conga drum graph, then every initial ideal graph H that it generates consists of all down-left components.*

Computing Regularity

Given a down-left graph, we can compute its induced matching number, μ, which is the size of the largest induced matching in the graph.

Theorem 5. $\mu(G(n, m)) = \min\{n - 1, m - 1\}$.

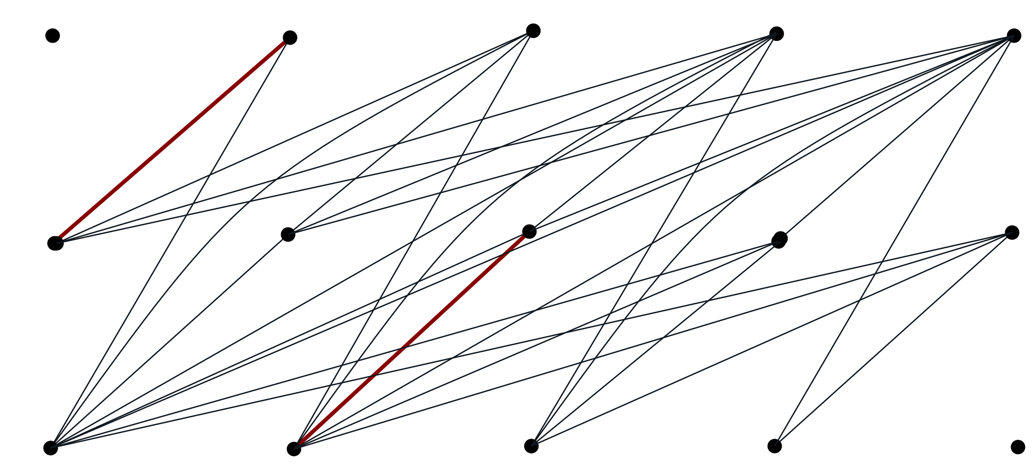

Fig. 6: $G(3, 5)$ with edges in red highlighting that $\mu(G(3, 5)) = 2$.

The induced matching number of a graph is the sum of the induced matching numbers of its components. In the cases highlighted by Theorem 4, we apply Theorem 5 to compute the regularity of $I(G)$.

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