Standing Waves of the Schrödinger Equation with Concentrated nonlinearity

Abba Ibrahim Ramadan Prof. Atanas Stefanov University of Kansas

Abstract

We study the concentrated NLS on $\mathbf{R}^n$, with power non-linearities, driven by the fractional Laplacian, $(-\Delta)^s, s > \frac{n}{2}$. We construct the solitary waves explicitly, in an optimal range of the parameters, so that they belong to the natural energy space H^s. Next, we provide a complete classification of their spectral stability. Finally, we show that the waves are non-degenerate and consequently orbitally stable, whenever they are spectrally stable.

Model

We studied the existence of standing wave solutions of the focusing NLS with concentrated non-linearity

$$\begin{cases} iu_t = ((-\Delta)^s - |u|^{2\sigma}\delta_0)u, \quad (t,x) \in \mathbf{R}\times\mathbf{R}^n \\ u(0,x) = u_0(x) \end{cases}. \tag{0.1}$$

Our definition of a solution is as follows: a continuous in x function u is a weak solution of (0.1), if it satisfies the inner product of (0.1) with all test functions ψ.

Our main interest in the model (0.1) are its solitary waves and their stability. More specifically, we consider solutions in the form $u = e^{i\omega t}\phi$, ϕ real-valued, which naturally satisfy the profile equation. This is again understood in the weak sense described above

$$(-\Delta)^s\phi + \omega\phi - |\phi(0)|^{2\sigma}\phi(0)\delta_0 = 0. \tag{0.2}$$

Main Result

Theorem 1. *(Existence standing waves of the concentrated NLS) Let $\omega > 0, s > \frac{n}{2}$ and $\sigma > 0$. Then, the function ϕ, with*

$$\hat{\phi}_\omega(\xi) = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi\right)^{-(1+\frac{1}{2\sigma})} \frac{1}{(2\pi|\xi|)^{2s}+\omega}.$$

is a solution of (0.2).

Theorem 2. *Let $n \geq 1$, $s > \frac{n}{2}$ and $\omega > 0$. Then, the waves $e^{i\omega t}\phi_\omega$ are spectrally stable if and only if*

$$0 < \sigma < \frac{2s}{n} - 1.$$

Theorem 3. *Let $n \geq 1$, $\omega > 0$, $s > \frac{n}{2}$, $0 < \sigma < \frac{2s}{n} - 1$. In addition, assume continuous dependence on initial data and globality of the solutions close to ϕ_ω, as outlined above. Then, the solitons $e^{i\omega t}\phi_\omega$ is orbitally stable.*

Existence of Solitary Waves

The proof of the theorem (1) is based on the calculus of variation technique. For the variational setting we approximated the 0. That is, V is non-negative, radial, smooth and supported on the unit ball $\mathbf{B} \subset \mathbf{R}^n$, with $\int_{\mathbf{B}} V(x)dx = 1$

Setting the variational problem

Let $\omega, \sigma > 0$. For a radial function $V : \mathbf{R}^n \to \mathbf{R}$ as above and $N >> 1$, consider the functional

$$I_{\omega,N}[u] = \frac{\int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|^2 dx + \omega\int_{\mathbb{R}^n} u^2 dx}{\left(\int_{\mathbb{R}^n} N^n V(Nx)|u|^{2\sigma+2}dx\right)^{\frac{1}{\sigma+1}}},$$

and the corresponding unconstrained variational problem $I_{\omega,N}[u] \to \min$. Clearly, $I_{\omega,N}[u] > 0$, so its optimal value is well-defined

$$m_N(\omega) := \inf_{u\in\mathcal{S}, u\neq 0} I_{\omega,N}[u].$$

Proposition 1. *Let $s > \frac{n}{2}$. Then the unconstrained minimization problem*

$$I_{\omega,N}[u] \to \min \tag{0.3}$$

has a real-valued solution $\phi_N \in H^s(\mathbf{R}^n) \cap L^\infty$, in particular $m_N(\omega) > 0$. Moreover, ϕ_N may be chosen to satisfy

$$N^n \int_{\mathbf{R}^n} V(Nx)|\phi_N(x)|^{2\sigma+2}dx = 1.$$

Finally, ϕ_N satisfies the Euler-Lagrange equation

$$(-\Delta)^s\phi_N + \omega\phi_N - m_N(\omega)N^nV(Nx)|\phi_N|^{2\sigma}\phi_N = 0 \tag{0.4}$$

in distributional sense.

Lemma 1. *There exists constants $C_1(\omega), C_2(\omega)$, but independent on N, so that*

$$C_1(\omega) \leq m_N(\omega) \leq C_2(\omega).$$

Furthermore, the sequence $\{\phi_N\}_{N=1}^\infty$, is a pre-compact in every set of the form $C(K)$, where K is a compact subset of $\mathbf{R}^n$.

Lemma 2. *The non-trivial solutions to* (0.2)*, with $\phi(0) > 0$ are given by*

$$\hat{\phi}(\xi) = \left(\int_{\mathbf{R}^n} \frac{1}{(2\pi|\xi|)^{2s}+\omega} d\xi\right)^{-(1+\frac{1}{2\sigma})} \frac{1}{(2\pi|\xi|)^{2s}+\omega}.$$

Eigenvalue Problem

To define the domain of the linearized operator, introduce the Green's function of the operator $(-\Delta)^s + \lambda$, namely the function $\mathcal{G}_s^\lambda$, so that

$$((-\Delta)^s + \lambda)\mathcal{G}_s^\lambda = \delta_0.$$

By taking the Fourier transform, we can write the following formula for $\mathcal{G}_s^\lambda$

$$\widehat{\mathcal{G}_s^\lambda}(\xi) = \frac{1}{(2\pi|\xi|)^{2s}+\lambda}.$$

Clearly, since $s > \frac{n}{2}$, $\mathcal{G}_s^\lambda \in H^s(\mathbf{R}^n) \subset C(\mathbf{R}^n)$. Introduce the domain of the operator $\mathcal{L}_c$ as

$$D(\mathcal{L}_c) = \{\psi \in H^s(\mathbf{R}^n) : \psi = g + c\psi(0)\mathcal{G}_s^\lambda, g \in H^{2s}(\mathbf{R}^n)\}. \tag{0.5}$$

With this domain, its action is defined as

$$\mathcal{L}_c\psi := ((-\Delta)^s + \lambda)g. \tag{0.6}$$

If we linearize around the solitary waves and ignore quadratic and higher order contributions, we obtain a linear system, whose spectral information plays a part in the dynamics. To that end, we take $u = e^{i\omega t}(\phi + v)$ and plug it in (0.1), ignoring any $O(v^2)$ term, utilizing (0.2) and setting $(v_1, v_2) := (\Re v, \Im v) = v$, we obtain

$$\begin{pmatrix} \Re v \\ \Im v \end{pmatrix}_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix}\begin{pmatrix} \Re v \\ \Im v \end{pmatrix},$$

where the following fractional Schrödinger operators are introduced

$$\begin{aligned} \mathcal{L}_+ &= (-\Delta)^s + \omega - (2\sigma+1)|\phi(0)|^{2\sigma}\delta_0, \\ \mathcal{L}_- &= (-\Delta)^s + \omega - |\phi(0)|^{2\sigma}\delta_0. \end{aligned}$$

$$\mathcal{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathcal{L} := \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix},$$

and the assignment $\begin{pmatrix} \Re v \\ \Im v \end{pmatrix} \to e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =: e^{\lambda t}\vec{v}$, we obtain the following time-independent linearized eigenvalue problem

$$\mathcal{J}\mathcal{L}\vec{v} = \lambda\vec{v}. \tag{0.7}$$

Lemma 3. *The closed symmetric operator $\mathcal{L}_c$, with domain given in* (0.5) *and whose action is defined in* (0.6)*, is self-adjoint.*

Stability

Proposition 2. *Let $s > \frac{n}{2}, \omega > 0, \sigma > 0$. Then,*

- *$\mathcal{L}_- \geq 0$, 0 is a simple eigenvalue, with eigenfunction $\mathcal{G}_s^\omega$ and*

  $$\sigma(\mathcal{L}_-) \setminus \{0\} \subset [\omega, \infty).$$

  Also, $\mathcal{L}_-|_{\{\mathcal{G}_s^\omega\}^\perp} \geq \omega$.

- *$\mathcal{L}_+$ has a simple negative eigenvalue, with an eigenfunction Ψ_0. Also,*

  $$\mathcal{L}_+|_{\{\Psi_0\}^\perp} \geq \omega > 0.$$

We need to apply the instability index counting theory to the eigenvalue problem (0.7). Since, $Ker(\mathcal{L}) = span\begin{pmatrix} \phi_\omega \\ 0 \end{pmatrix}$. It follows that the spectral stability of it is equivalent to the condition $\langle\mathcal{L}_+^{-1}\phi_\omega, \phi_\omega\rangle < 0$.
Since, $\phi_\omega = c\mathcal{G}_s^\omega$, it suffices to compute $\langle\mathcal{L}_+^{-1}\mathcal{G}_s^\omega, \mathcal{G}_s^\omega\rangle$, and this proves Theorem 2.

Proposition 3. *Let $s > \frac{n}{2}, \omega > 0$, $\langle\mathcal{L}_+^{-1}\phi_\omega, \phi_\omega\rangle < 0$. Then, the operator $\mathcal{L}_+$ is coercive on $\{\phi_\omega\}^\perp$. That is, there exists $\delta > 0$, so that for all*

$$\langle\mathcal{L}_+\Psi, \Psi\rangle \geq \delta\|\Psi\|_{H^s}^2, \quad \forall\Psi \perp \phi_\omega. \tag{0.8}$$

Clearly, the orbital stability is only expected to hold for the case $\sigma < \frac{2s}{n} - 1$, so we assume that henceforth. We make the following **key assumptions**:

1. The solution map $g \to u_g$ has **continuous dependence on initial data property in a neighborhood of ϕ.**
2. **All initial data, sufficiently close to ϕ_ω in H^s norm, generates a global in time solution** u_g of (0.1). In addition, the L^2 norm and the Hamiltonian for these solutions are conserved.

Proposition 4. *Let $\omega > 0$, $n \geq 1, s > \frac{n}{2}$, $0 < \sigma < \frac{2s}{n} - 1$ and the key assumptions (1), (2) are satisfied. Then $e^{i\omega t}\phi_\omega$ is orbitally stable solution of* (0.1).

References

[1] Abba Ramadan, Atanas G Stefanov. On the standing waves of the Schrödinger equation with concentrated nonlinearity *Anal.Math.Phys.* ,11, 136 (2021)

[2] R. Carlone, D. Finco, L. Tentarelli, *Nonlinear singular perturbations of the fractional Schrödinger equation in dimension one, Nonlinearity*, **32**, (2019), p. 3112–3143.