Gauss-like Exponential Sums from Whittaker Coefficients

Swapnil Garg[1], Aidan Kelley[2], Frank Lu[3], and Siki Wang[4]. Advisor: Ben Brubaker[5]
[1]MIT [2]Washington University in St. Louis [3]Princeton University [4]Claremont McKenna College [5]University of Minnesota

Gauss Sums

Definition 1. *Given $t \in \{1,2\}$, $a \in \mathbb{Z}$, p prime, and $\ell \geq 0$, a Gauss Sum is a weighted sum over roots of unity, defined as*

$$g_t(a, p^l) = \sum_{c \bmod p^l} \left(\frac{c}{p}\right)_2^{lt} \exp\left(2\pi i \frac{ac}{p^l}\right)$$

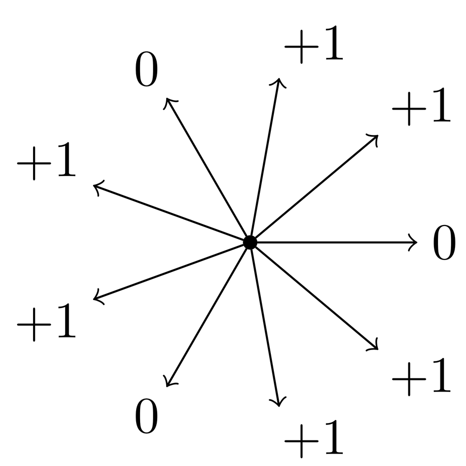
Definition 2. *For $c \in \mathbb{Z}$, p prime, the quadratic residue symbol is*

$$\left(\frac{c}{p}\right)_2 = \begin{cases} 0 & (c,p) \neq 1 \\ 1 & \text{there exists } b, c \equiv b^2 \bmod p \\ -1 & \text{there does not exist } b, c \equiv b^2 \bmod p \end{cases}$$

Proposition 3. *For a, p relatively prime, $g_t(a, p^l) = \left(\frac{a}{p}\right)_2^{-lt} g_t(1, p^l)$.*

Proposition 4. *For a, p relatively prime, $l \geq 2$, $g_t(a, p^l) = 0$.*

$g_1(1, 3^2) = 0$. The cancellation of this sum is visualized to the right.



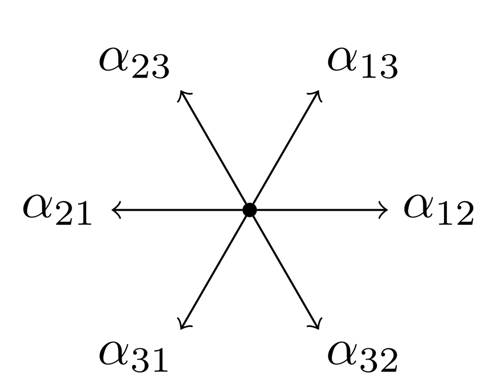
Root Systems

A root system is a highly symmetric set of vectors which is important in understanding the representation theory of matrix groups.

Definition 5. *The root system Φ_n of type A_n is the collection of vectors*

$$\Phi_n = \{\alpha_{ij} : 1 \leq i, j \leq n+1, i \neq j\} \subset \mathbb{R}^{n+1}, \ \alpha_{ij} = e_i - e_j$$

Φ_2 spans a space of rank 2, and the projection of Φ_2 onto that space looks like



Definition 6. *We define $\Phi_n^+ = \{\alpha_{i,j} : i < j\} \subset \Phi_n$ to be the positive roots.*

We can associate the root α_{ij} of Φ_n^+ to the ith row and jth column in a $n+1 \times n+1$ matrix.

$$\begin{bmatrix} * & \alpha_{12} & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ & * & \gamma_8 & \gamma_7 & \gamma_6 & \gamma_5 \\ & & * & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & * & \alpha_{45} & \alpha_{46} \\ & & & & * & \alpha_{56} \\ & & & & & * \end{bmatrix}$$

Definition 7. *We call the roots in in Φ_n^+ that cannot be written as positive linear combinations of other roots the simple roots*

In our case, these are $\alpha_{i,i+1}$, $1 \leq i \leq n$.

Definition 8. *We can define a partial order on Φ_n^+ by defining $\alpha > \beta$ if $\alpha - \beta$ is a positive linear combination of simple roots.*

Whittaker Coefficients of Metaplectic Eisenstein Series

Our goal for this project is to compute the Whittaker coefficients for parabolic Eisenstein Series on a metaplectic cover of $GL_6(F)$, F a number field. Whittaker coefficients are multiple Dirichlet series with nice analytic properties. By a Conjecture of Bump, this Dirichlet Series should match one of Chinta with similar analytic properties [2]. In this project, we compute this Dirichlet series in a specific way to try to match the series of Chinta.

The H sum

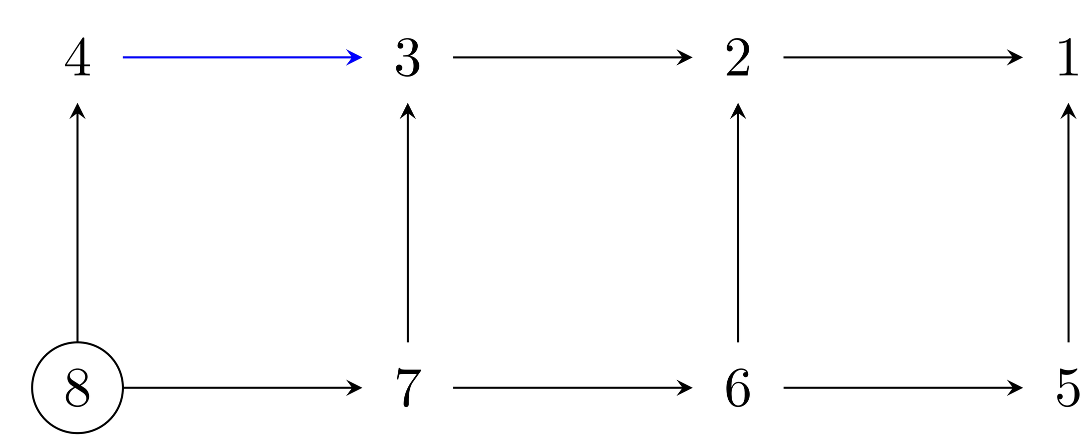
This computation involves a large integral over the group. Through the methods of [1], we can parameterize this integral using the root system data. The central piece of the computation then becomes the following sum, which was defined in terms of inner products and orderings between the roots.

Definition 9. *Let $\mathbf{d} = (d_1, \dots, d_8)$ with each $d_i = p^{l_i}$. The H-sum is*

$$H(\mathbf{d}) = \sum_{c_i \bmod D_i} \exp\left(2\pi i \left(-\frac{b_5c_1d_6d_7d_8}{d_1d_2d_3d_4} - \frac{b_6c_2d_7d_8}{d_2d_3d_4} - \frac{b_7c_3d_8}{d_3d_4} - \frac{b_8c_4}{d_4} + \frac{c_8}{d_8} + \frac{b_4c_3d_8}{d_3d_7} + \frac{b_8c_7}{d_7} + \frac{b_3c_2d_7}{d_2d_6} + \frac{b_7c_6}{d_6} + \frac{b_2c_1d_6}{d_1d_5} + \frac{b_6c_5}{d_5}\right)\right) \prod_{k=1}^{8} \left(\frac{c_k}{p}\right)_2^{l_i}$$

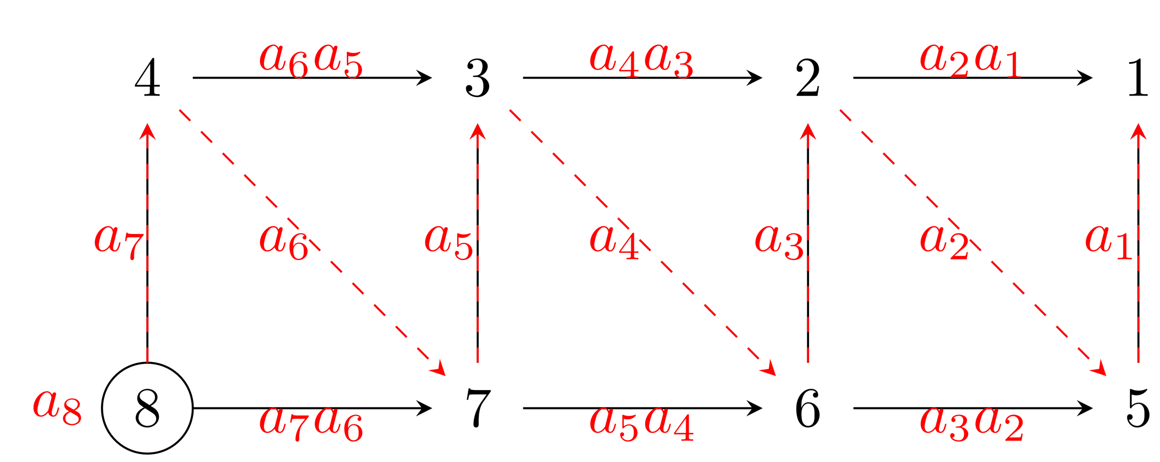
where $b_ic_i \equiv 1 \bmod d_i$ and D_is are functions of d_is.

Proposition 10 (Dependency Graph). *There is a b_ic_j term in the exponent of the sum $\iff$ there is an edge $i \to j$ in the graph.*



We circle 8 to note the additional $\frac{c_8}{d_8}$ term.

We notice this is a sum over edges in this graph, so we can simplify the sum structure by reindexing to variables $a_1, \dots, a_8$ such that the edges become the summation variables.



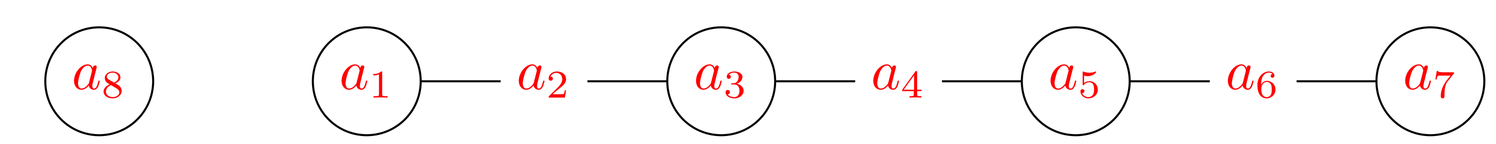
This diagram means $a_1 = b_5c_1, a_2 = b_2c_5$, and as an example we see $b_2c_1 = a_2a_1$ since $a_2a_1 = b_2c_5b_5c_1 = b_2c_1$ (proper care is needed to justify $c_5b_5 = 1$).

The Reparameterized Sum

After reparameterization, we get the sum

$$H(\mathbf{d}) = \sum_{a_i} \exp\left(2\pi i \left(+\frac{a_8}{d_8} - \frac{a_1d_6d_7d_8}{d_1d_2d_3d_4} + \frac{a_1a_2d_6}{d_1d_5} + \frac{a_2a_3}{d_5} - \frac{a_3d_7d_8}{d_2d_3d_4} + \frac{a_3a_4d_7}{d_2d_6} + \frac{a_4a_5}{d_6} - \frac{a_5d_8}{d_3d_4} + \frac{a_5a_6d_8}{d_3d_7} + \frac{a_6a_7}{d_7} - \frac{a_7}{d_4}\right)\right) \prod_{k=1}^{8} \left(\frac{a_k}{p}\right)_2^{l_i'},$$

where the l_i' are a function of the original l_is. In a similar way, we can associate the following graph to this sum:



Notice that a_8 actually factors out as a Gauss sum, so we see

$$H(\mathbf{d}) = g_{l_8'}(1, p^{l_8}) \cdot H'(\mathbf{d}).$$

Results

We make the following observation: Say a term has denominator p^l

- If $l \leq 0$, the term vanishes
- If $l \geq 2$, then using a technique similar to Proposition 4, the sum cancels or becomes simpler.

Then, we see the hardest case to solve is when the denominators are all p.

Theorem 11 (GKLS). *We can solve these $H'(\mathbf{d})$ sums inductively.*

The result of our work is an inductive algorithm that can compute the sums explicitly in terms of p. The algorithm runs in three stages

1. In the "simple" case, pull a_1, a_2 out using Propositions 3 and 4.
2. Isolate and solve the "hard" parts of the sum.

The remaining work is to use this computation to finish the full evaluation of the Whittaker coefficient to compare against Chinta's series.

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References

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