THE RESTRICTED LIE ALGEBRA STRUCTURE ON THE BAR SPECTRAL SEQUENCE OF AN ITERATED LOOP SPACE

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Abstract

There is a rich algebraic structure in the mod 2 homology of the iterated loop space $H_*(\Omega^n X;\mathbb{F}_2)$. It admits Lie bracket that is compatible with the Dyer-Lashof operations $Q_0, Q_1, \ldots, Q_{n-1}$. Furthermore, the top Dyer-Lashof operation Q_{n-1} is a restriction for the Browder bracket. Ni proved that the Browder bracket on $H_*(\Omega^n X)$ converges to the bracket on $H_*(\Omega^{n-1}X)$ in the bar spectral sequence. Our goal is to use the bar spectral sequence to relate the restricted Lie algebra structure given by the top operation on $H_*(\Omega^n X;\mathbb{F}_2)$ to that of $H_*(\Omega^{n-1}X;\mathbb{F}_2)$.

Background

1. Preliminary Structure

For a space X with basepoint $*$, define the n-fold *loop space* $\Omega^n X$ as the set of functions $\gamma : I^n \to X$ such that $\gamma(\partial I^n) = *$ equipped with the compact open topology. Our results are all over the field $\mathbb{F}_2$, so we will omit the coefficient going forward. The homology of the n-fold loop space is a Poisson-Hopf algebra with the Browder bracket (c.f. [2])

$$[-,-] : H_p(\Omega^2 X) \otimes H_q(\Omega^2 X) \to H_{p+q+n-1}(\Omega^2 X),$$

known as and a restriction map Q_{n-1} satisfying the following relations:

- Antisymmetry: $[x, y] = [y, x]$;
- Poisson Identity: $[x, yz] = [x, y]z + y[x, z]$;
- Jacobi Law: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$;
- Top Additivity: $Q_{n-1}(x + y) = Q_{n-1}(x) + Q_{n-1}(y) + [x, y]$;
- Adjoint Identity: $[Q_{n-1}(x), y] = [x, [x, y]]$.

2. The Normalized Bar Construction Let $\mathcal{A} = H_*(\Omega^n X)$, and let $\varepsilon : \mathcal{A} \to H_*(*) \cong \mathbb{F}_2$ be the augmentation map. The normalized bar construction $B_{*,*}(\mathcal{A})$ is as follows. Denote by $\overline{\mathcal{A}}$ the kernel of ε. Set

$$B_{s,*} = \overline{\mathcal{A}} \otimes \cdots \otimes \overline{\mathcal{A}},$$

where we repeat $\overline{\mathcal{A}}$ s times. Here, $B_{0,*} = k$. The elements of $B_{s,t}$ are defined to be those elements of $B_{s,*}$ with *internal degree* t. Typically, an element $x_1 \otimes \cdots \otimes x_s \in B_{s,*}$ is written as $[x_1|\cdots|x_s]$. There is an *internal differential* d of bidegree $(0, -1)$ and an *external differential* δ of bidegree $(-1, 0)$ given by

$$d[x_1|\cdots|x_s] = \sum_{i=1}^{s}[x_1|\cdots|d_{\mathcal{A}}x_i|\cdots|x_s], \text{ and } \delta[x_1|\cdots|x_s] = \sum_{i=1}^{s-1}[x_1|\cdots|x_i x_{i+1}|\cdots|x_s],$$

respectively. Finally, the *total differential* is $D = d + \delta$.

We define the comultiplication $\Delta : B_{*,*}(\mathcal{A}) \to B_{*,*}(\mathcal{A}) \otimes B_{*,*}(\mathcal{A})$ via

$$\Delta([x_1|\cdots|x_s]) = \sum_{i=0}^{s}[x_1|\cdots|x_i] \otimes [x_{i+1}|\cdots|x_s],$$

which provides a coalgebra structure (here we set $[] = 1 \in B_{0,*}$).

Definition. For two nonnegative integers p and q, a (p, q)-*shuffle* is a permutation $\varphi \in \Sigma_{[p+q]}$ satisfying $\varphi(a) < \varphi(b)$ if $1 \le a < b \le p$ or if $p + 1 \le a < b \le p + q$. Note that there are $\binom{p+q}{p}$ such permutations.

Given two elements $[x_1|\cdots|x_p]$ and $[y_1|\cdots|\alpha_q]$ in the bar construction, there is a *shuffle product* $B_{*,*}(\mathcal{A}) \otimes B_{*,*}(\mathcal{A}) \to B_{*,*}(\mathcal{A} \otimes \mathcal{A})$ given by the Eilenberg-Zilber map.

3. The Bar Spectral Sequence

We follow the exposition in [3]. Define the total complex on the bar construction to be

$$(\mathrm{tot}\, B_{*,*}(\mathcal{A}))_n = \bigoplus_{p+q=n} B_{p,q}(\mathcal{A})$$

with differential D. The homology of this chain complex with coefficients in a field k is

$$H_*(\mathrm{tot}\, B_{*,*}(\mathcal{A}); k) = \mathrm{Tor}_*^{\mathcal{A}}(k, k).$$

We define a filtration on each $(\mathrm{tot}\, B_{*,*}(\mathcal{A}))_n$ by taking

$$F_s(\mathrm{tot}\, B_{*,*}(\mathcal{A}))_n = \bigoplus_{\substack{p+q=n \\ p \le s}} B_{p,q}(\mathcal{A}).$$

The associated graded pieces of the filtration are

$$E^0_{s,t} = F_s(\mathrm{tot}\, B_{*,*}(\mathcal{A}))_{s+t}/F_{s-1}(\mathrm{tot}\, B_{*,*}(\mathcal{A}))_{s+t} = B_{s,t}(\mathcal{A}).$$

The differentials on the E^0 and E^1-page are $d_0 = d$ and $d_1 = \delta$. Since $\mathcal{A}$ is a k-algebra

$$H_*(B_{s,*}(\mathcal{A})) = H_*(\underbrace{\overline{\mathcal{A}} \otimes \cdots \otimes \overline{\mathcal{A}}}_{s \text{ times}}) \cong \underbrace{\overline{H_*(\mathcal{A})} \otimes \cdots \otimes \overline{H_*(\mathcal{A})}}_{s \text{ times}} \cong B_{s,*}(H_*(\mathcal{A})),$$

and so the E^1-page is the bar construction of the homology of $\mathcal{A}$ with a trivial internal differential. This gives rise to a strongly convergent homological spectral sequence

$$E^2_{*,*} \cong \mathrm{Tor}_*^{H_*(\mathcal{A})}(k, k) \Rightarrow \mathrm{Tor}_*^{\mathcal{A}}(k, k). \tag{1}$$

Here we are interested in the case $\mathcal{A} = C_*(\Omega^{n-1}X)$. It is well known that there is a quasi-isomorphism

$$\mathrm{tot}\, B_{*,*}(C_*(\Omega^n X)) \xrightarrow{\simeq} C_*(\Omega^{n-1}X).$$

Passing to homology yields an isomorphism of Hopf algebras

$$\mathrm{Tor}_*^{C_*(\Omega^n X)}(k, k) \cong H_*(\Omega^{n-1}X).$$

Clark [1] proves that 1 is a spectral sequence of Hopf algebras. To sum up, the spectral sequence relates the bar construction of $H_*(\Omega^n X)$ to the homology of $\Omega^{n-1}X$.

Our Results

Theorem. Let $x = [x_1|\cdots|x_s]$. If $s = 1$, we set $\xi(x) = [Q_{n-1}(x_1)]$. For $s > 1$, we take

$$\xi(x) = \sum_{\substack{(s,s)-\text{shuffles } \varphi \\ \text{with } \varphi^{-1}(1) = 1}} \sum_{\substack{\varphi^{-1}(i) \le s \\ \varphi^{-1}(i+1) > s}} [z_{\varphi^{-1}(1)}|\cdots|[z_{\varphi^{-1}(i)}, z_{\varphi^{-1}(i+1)}]|\cdots|z_{\varphi^{-1}(2s)}],$$

where

$$z_i = \begin{cases} x_i & \text{if } i \le s \\ x_{i-s} & \text{if } i > s \end{cases}.$$

We extend ξ to the entire bar construction via top additivity. Then, the operation $\xi : B_{s,t}(H_*(\Omega^n X)) \to B_{2s-1.2t+1}(H_*(\Omega^n X))$ is a restriction for the bracket:

- $\xi(x + y) = \xi(x) + \xi(y) + [x, y]$;
- $[x, \xi y] = [y, [x, y]]$;
- $\delta\xi x = [x, \delta x]$ and $dx = [x, dx]$

In particular, the bar construction $B_{*,*}(H_*(\Omega^n X))$ is a restricted Lie algebra.

Sketch of Proof:

- This follows by definition, since we extend ξ to the bar construction via top additivity.
- This holds when the simplicial degree of y is 1 by the adjoint identity. For higher degrees, the identity succumbs to a combinatorial proof realizing each term in the formula for $\xi(s)$ as a path from $(0, 0)$ to (s, s) with a fixed first step.
- The first identity succumbs to a combinatorial argument similar to the one above. The second identity is true, since it holds in the homology for the internal differential.

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References

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