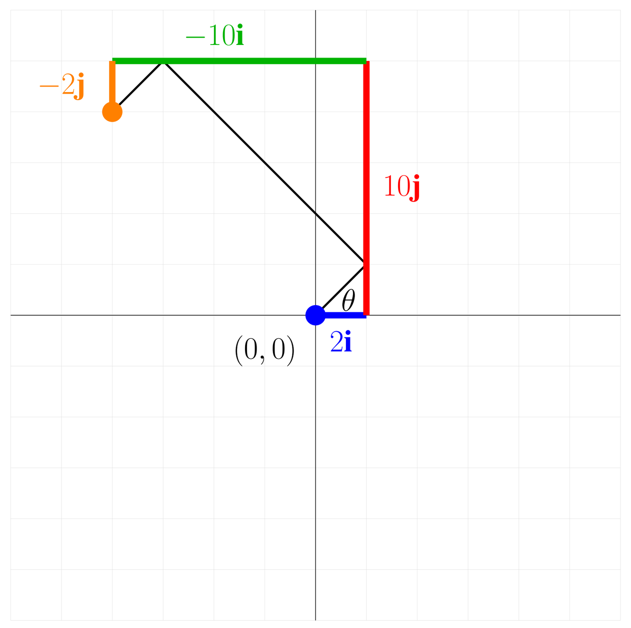
Lill Paths and Beloch Squares

Hadley Vaughn – Mentor: Dr. Jason Callahan

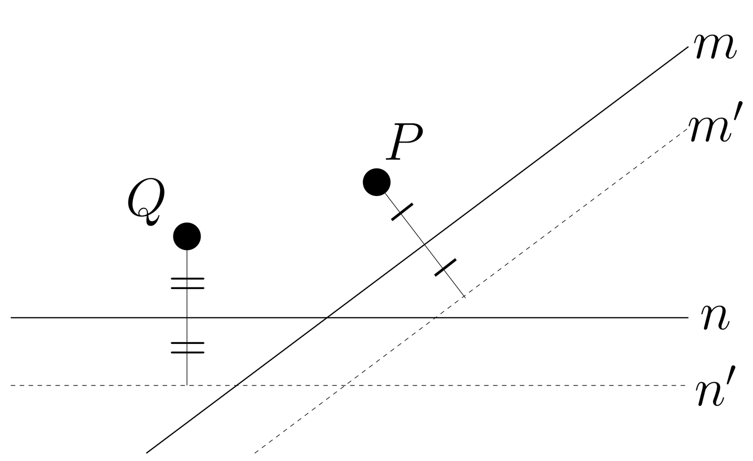
Lill's Method



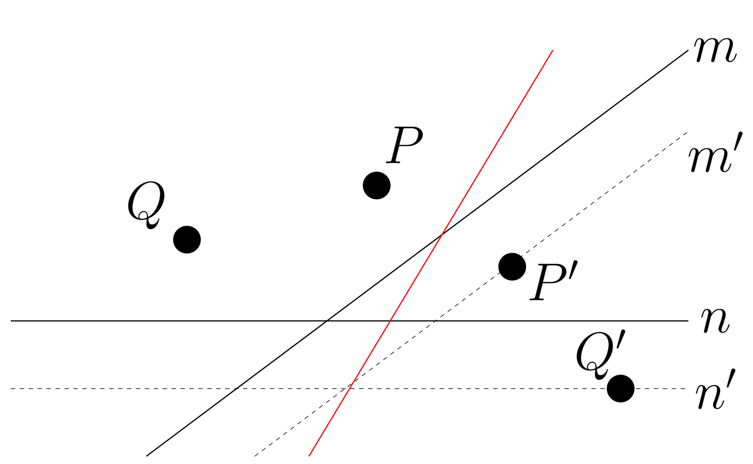
$p(x) = 2x^3 + 10x^2 + 10x + 2$

Lill's method to find real polynomial roots can be used for any polynomial of the form $p(x) = a_r x^r + a_{r-1}x^{r-1} + \cdots + a_0$. Starting at the origin, vectors in the repeating pattern of $\mathbf{i}, \mathbf{j}, -\mathbf{i}, -\mathbf{j}$ scaled by the polynomial's coefficients beginning with a_n are drawn tip to tail until all coefficients of $p(x)$ are represented by the Lill path as shown to the left [3]. A path can be constructed by extending a line from the origin $\theta°$ above the horizontal that reflects off the Lill path at $90°$. This is called a θ-path where by varying θ, the end of the θ-path coincides with the end of the Lill path as shown above. When this occurs, the roots of the polynomial are equal to $-\tan\theta$. Any valid θ-path must have $r-1$ edges.

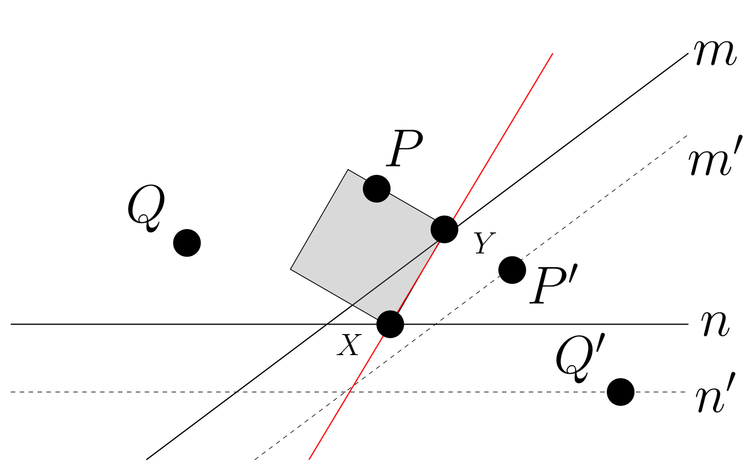
The Beloch Fold



The Beloch fold [2] begins with two points, P and Q. We construct lines m and n and lines m' and n' such that the perpendicular reflection of P over m and Q over n produce m' parallel to m and n' parallel to n.

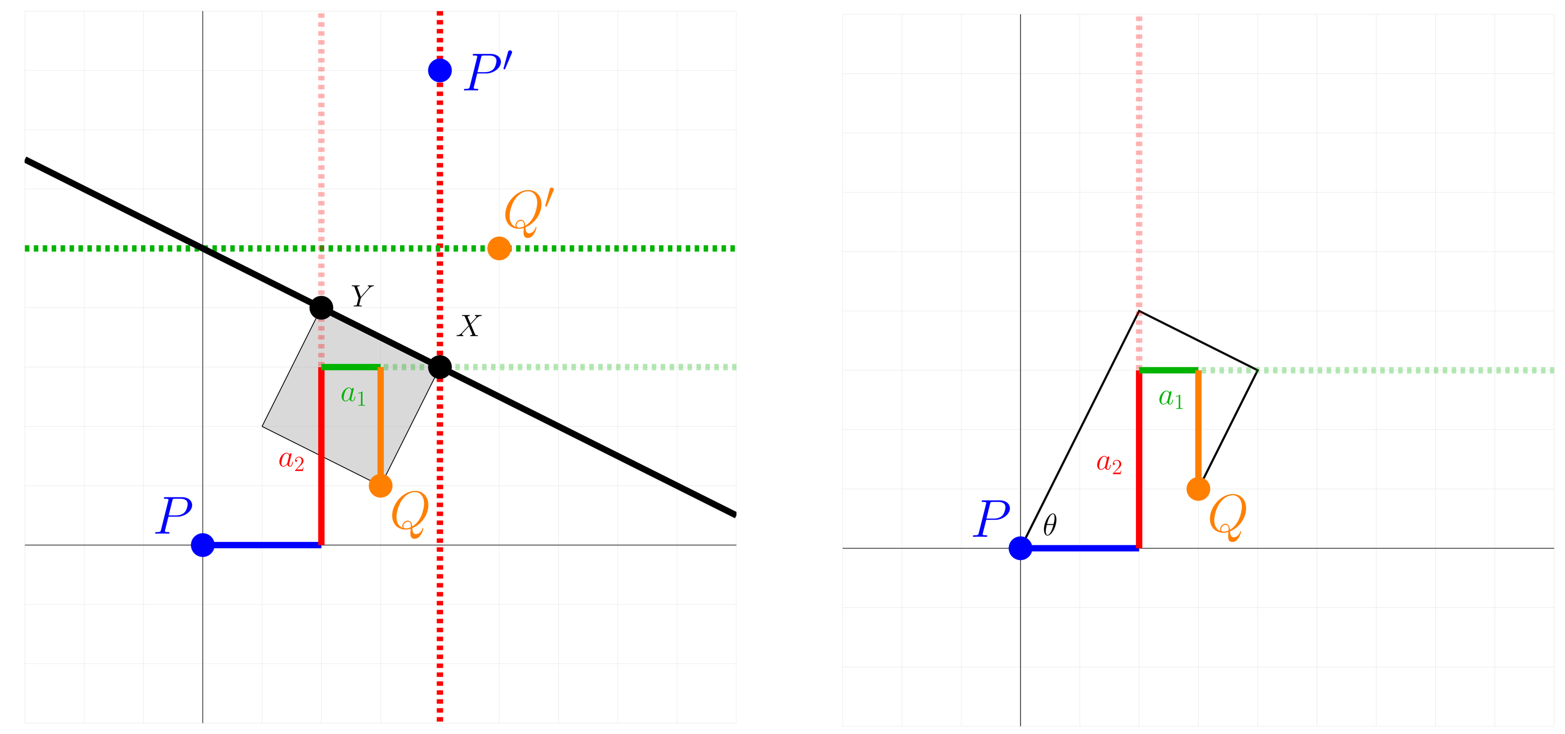


The fold line shown in red identifies P to a point P' on m' and Q to a point Q' on n'.



Where the fold line intersects n, we label the point X and where the fold line intersects m, we label Y. Using the distance between X and Y, we construct the shaded Beloch square. Dr. Beloch used this method to geometrically construct $\sqrt[3]{2}$.

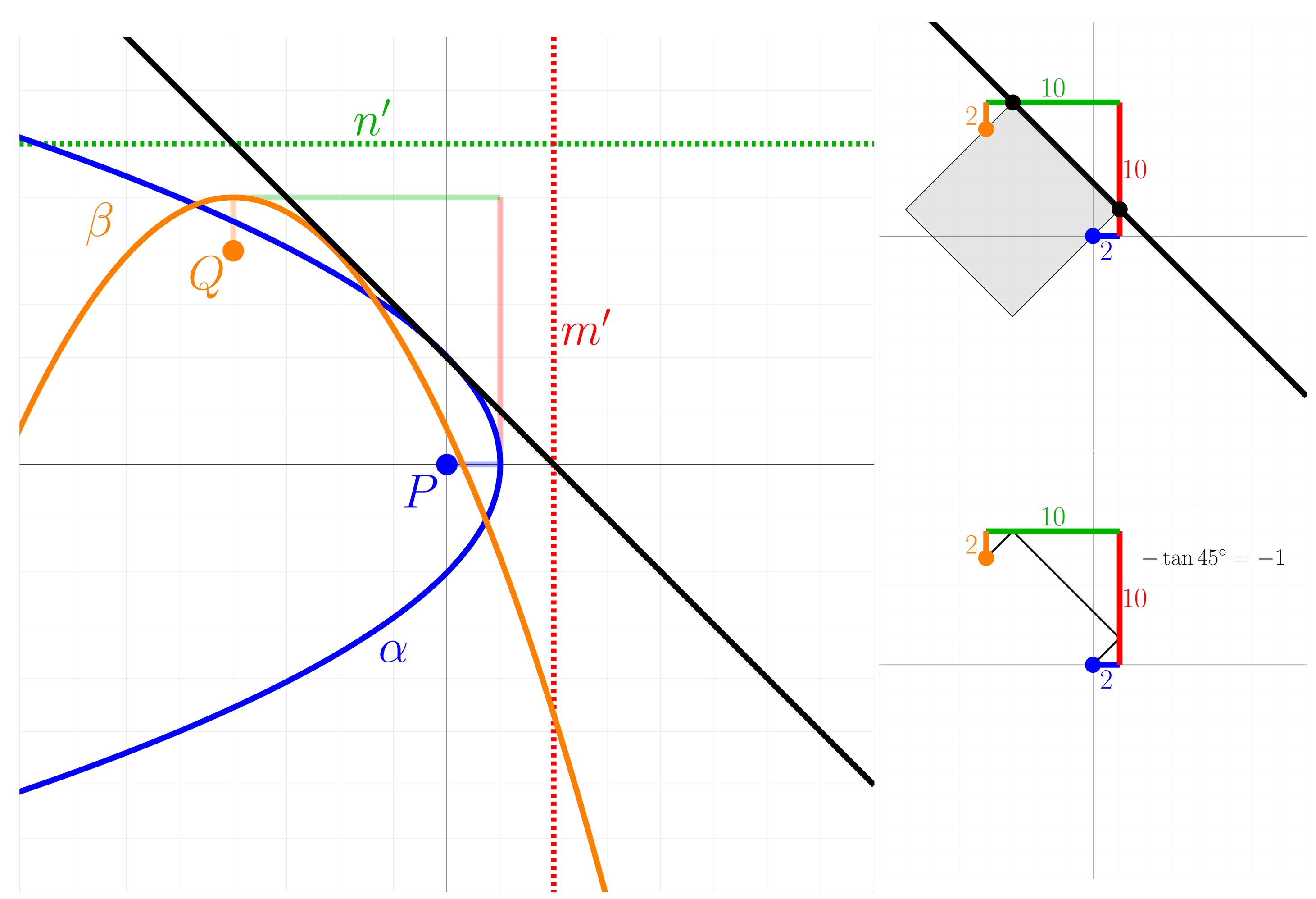
The Beloch Square



$p(x) = a_3x^3 + a_2x^2 - a_1x + a_0, \ a_i > 0$

Let $m = a_2$ and $n = a_1$ be the extended legs corresponding to the Lill path for $p(x)$. Let point P be the origin and point Q be the terminus of this Lill path as shown above. One can construct a Beloch square which guides the solving θ-path for $p(x)$.

Parabolic θ-Path Solutions



In [4], applications of parabolas to Beloch folds are discussed. We will apply this idea to Lill paths. Instead of varying θ to find the correct angle for a solving θ-path, we construct two parabolas α and β with foci P and Q and directrixes m' and n'. Where α and β share a tangent line, one can construct a Beloch fold, which will produce a Beloch square that guides the solving θ-paths of $p(x)$. After ample observation of this property, we have formulated the following conjecture.

Conjecture 1. *Given parabolas α and β with foci P and Q and directrixes m' and n', where P is the origin and Q is the terminus associated with a Lill path corresponding a cubic polynomial, all shared tangent lines between α and β produce Beloch squares which correspond to θ-paths that give the roots of $p(x)$ as $-\tan\theta$.*

Future Research

In the future we will try to use multiple Beloch squares to generate θ-paths for Lill paths constructed from polynomials of degree $r > 3$. Alperin and Lang demonstrate the ability to construct θ-paths for higher degree polynomials using two folds [1].

We hypothesize that this is equivalent to constructing a string of Beloch squares in succession with the new origin the endpoint of the previous Beloch square. Initial explorations of this method have yielded θ-paths with more than $r-1$ edges, which is not valid. We hypothesize that forcing each Beloch square to share an edge may produce θ paths with $r-1$ edges.

In [5], Lill's method is applied to complex polynomial roots.We would like to see if there is a way to find these complex roots using a parabolic approach as before and discover what changes might need to be made to adapt the parabolic approach to find complex roots. Finally, we would like to formally prove Conjecture 1.

Acknowledgements

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References

[1] Origami 4 : Fourth International Meeting of Origami Science, Mathematics, and Education / Robert J. Lang, editor. Natick, Mass, 2009. International Meeting of Origami Science, Mathematics, and Education Corporate Author, A.K. Peters.

[2] Thomas C. Hull. Solving Cubics With Creases: The Work of Beloch and Lill. *The American Mathematical Monthly*, 118(4):307–315, 2011.

[3] Dan Kalman and Mark Verdi. Polynomials with Closed Lill paths. *Mathematics Magazine*, 88(1):3–10, 2015.

[4] J. Lucero. Existence of a Solution for Beloch's Fold. *Mathematics Magazine*, 92:24 – 31, 2019.

[5] Serge Tabachnikov. Polynomials as Polygons. *The Mathematical Intelligencer*, 39, 03 2017.