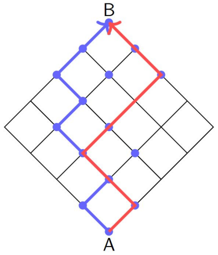
The Johnson-Leader-Russell Question on Square Posets

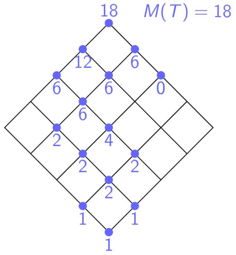
Abstract

We study the problem of finding the maximum number of maximal chains in a given size-k subset of a square poset $[n] \times [n]$. This was proposed by Johnson, Leader, and Russell but not yet solved. Kittipassorn had given a conjectural solution to the problem. We verify Kittipassorn's conjecture for $0 \le k \le 3n-2$ and solve a variant problem for the case $3n-1 \le k \le 4n-4$, which also supports the conjecture. For general k, we find that the optimal configuration is given by a 1-Lipschitz function. We also generalize the problem to rectangle posets and give a solution to one particular poset.

Question

- The "Johnson-Leader-Russell Question" was first proposed by J. Robert Johnson, Imre Leader, Paul Russell in 2013. [1]
- This question remains open.





Question (Johnson, Leader, and Russell)

Find a configuration T^* with k points such that $M(T^*)$ is maximized, i.e.

$$M(T^*) = \max_{T:|T|=k} M(T).$$

Kittipassorn's configuration

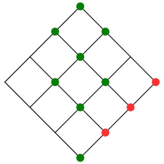
Question (Kittipassorn)

Find T^* with r_i points on the i-th level such that $M(T^*)$ is maximized.

Kittipassorn's configuration:

1. For each level, all the points are condensed in the middle.
2. Break the left-right symmetry, extra points are put on the right.

Denote such a configuration $T^*(r_1, r_2, \ldots, r_{2n-1})$.



Lemma 2.5 (Kittipassorn's lemma)

Suppose that non-negative $r_1, r_2, \ldots, r_{2n-1}$ are given. Then

$$\max_{T:\forall i, |T \cap L_i| = r_i} M(T) = M(T^*(r_1, r_2, \ldots, r_{2n-1})).$$

The case $2n$-$1 \le k \le 3n$-2

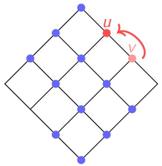
Proposition 3.3

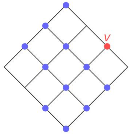
If $2n-1 \le k \le 3n-2$, then

$$\max_{T:|T|=k} M(T) = 2^{k-2n+1}.$$

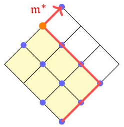
Investigation on general k

- After removing the "wasted" point, the number of paths remains the same.

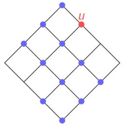




Remove a "wasted" point v



Find a path $\mathfrak{m}^*$ with the greatest **Area**



Add a "useful" point u opposite to a turning point of m^*

Theorem 3.11

For given $2n-1 \le k \le n^2$,

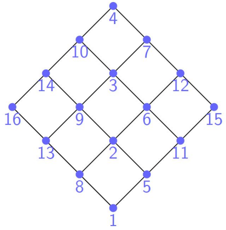
$$\max_{T:|T|=k} M(T) = \max M(T^*(f))$$

where f is taken over all 1-Lipschitz functions such that $\sum_{i=1}^{2n-1} f(i) = k$.

Conjectures

Conjecture 2.6 (Kittipassorn)

The points should be put from middle to sides, and from bottom to top.



- Conjectural solution for $3n-1 \le k \le 4n-4$

Definition

Let $\mathfrak{U}$ to be the set of configurations T with at most 2 points on each level, i.e.

$$|T \cap L_i| \le 2.$$

Conjecture 5.1

For $3n-1 \le k \le 4n-4$,

$$\max_{T:|T|=k} M(T) = \max_{T \in \mathfrak{U}} M(T).$$

Proposition 5.4

For $3n-1 \le k \le 4n-4$,

$$\max_{T \in \mathfrak{U}} M(T) = 2^{4n-k-4} F_{2k-6n+7}$$

where F_i denotes the i-th Fibonacci number.