An Exploration of Odd Prime Graph Labelings

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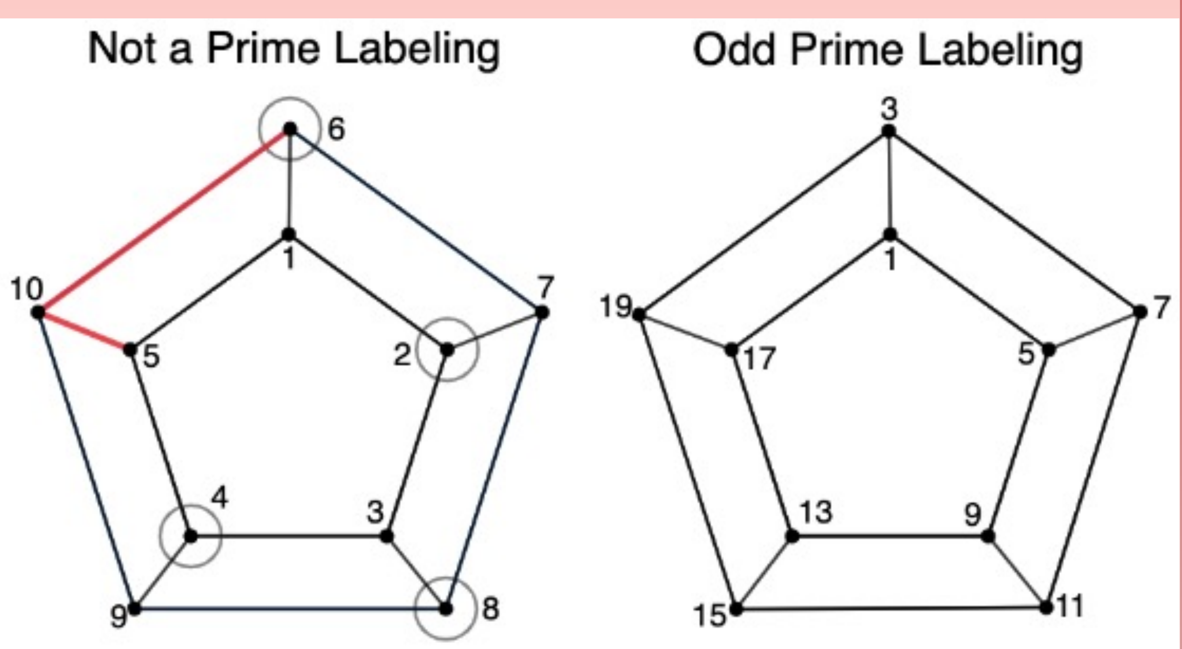
Abstract

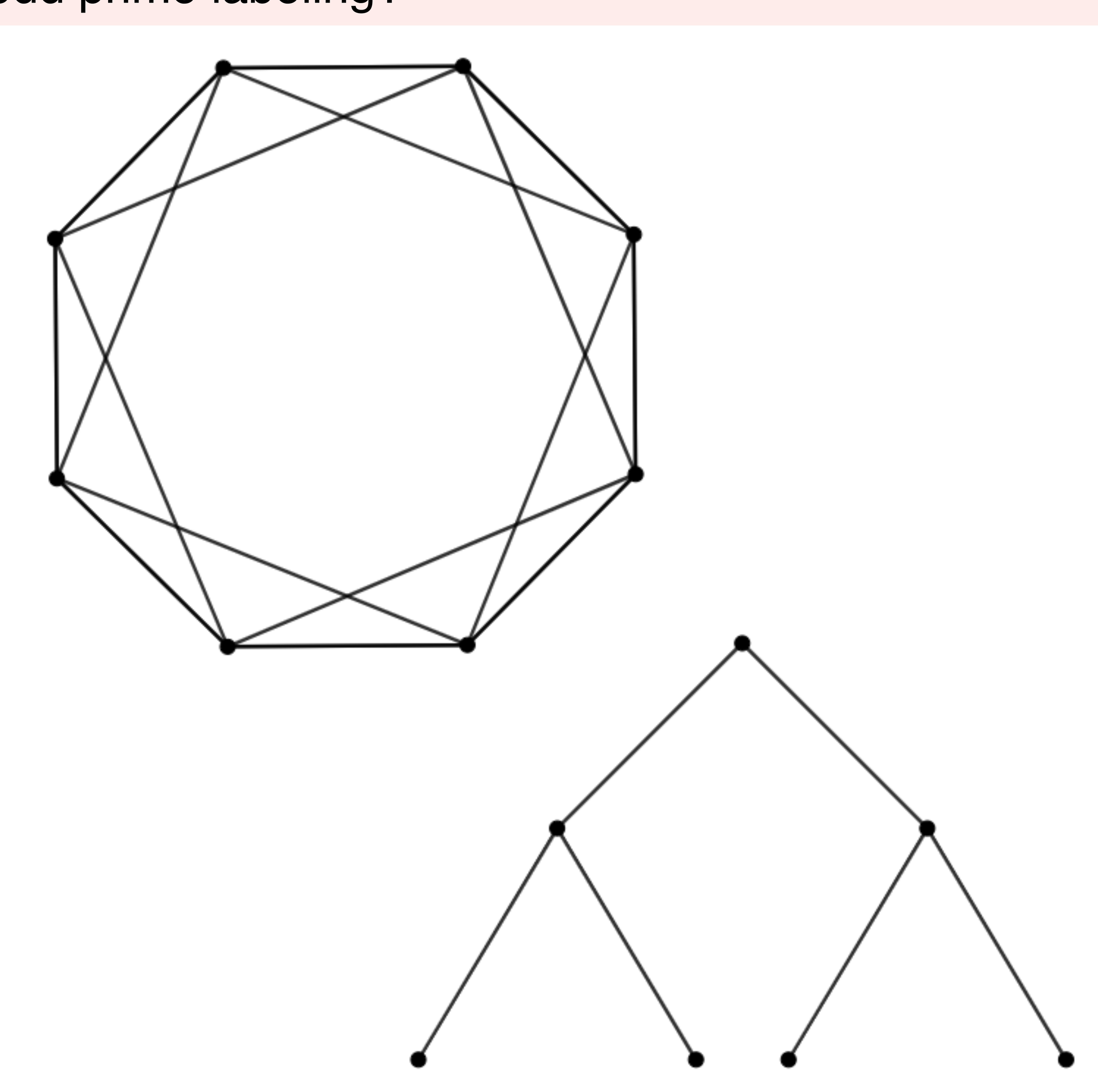
An odd prime labeling is a variation of a prime labeling in which the vertices of a graph of order n are labeled with the distinct odd integers 1 to 2n-1 so that the labels of adjacent vertices are relatively prime. This paper investigates many different classes of graphs including stacked prisms, binary trees and caterpillars, and uses various methods to construct odd prime labelings.

Definition and Examples

- G is a simple graph of order n (with order being the number of vertices)
- *Prime labeling* – an assignment of integers $1, \dots, n$ to the vertices of G so that labels of adjacent vertices are relatively prime
- *Odd Prime Labeling* — an assignment of the odd integers $1, \dots, 2n-1$ such that adjacent labels of vertices are relatively prime
- Note that odd numbers separated by powers of 2 are always relatively prime



Question: Which of the following two graphs has an odd prime labeling?



Combinations of Cycles

Disjoint Union of Cycles

- **Theorem 1**: All disjoint unions of cycles, $\bigcup_{i=1}^{m} C_{n_i}$ are odd prime for any lengths n_i and any number of cycles m. For each cycle of length n:
- If n is even, label the sequence of vertices:

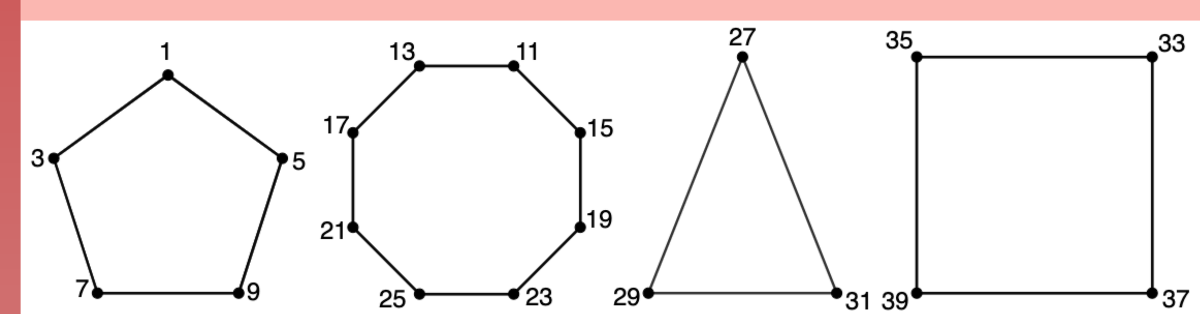
$$v_1, v_n, v_2, v_{n-1}, v_3, v_{n-2}, \dots, v_{\frac{n}{2}+1}, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}$$

with the labels $m, m+2, m+4, \dots, m+2n-2$

- If n is odd, label the sequence of vertices:

$$v_1, v_n, v_2, v_{n-1}, v_3, v_{n-2}, \dots, v_{\frac{n+1}{2}-1}, v_{\frac{n+1}{2}+1}, v_{\frac{n+1}{2}}$$

with the labels $m, m+2, m+4, \dots, m+2n-2$



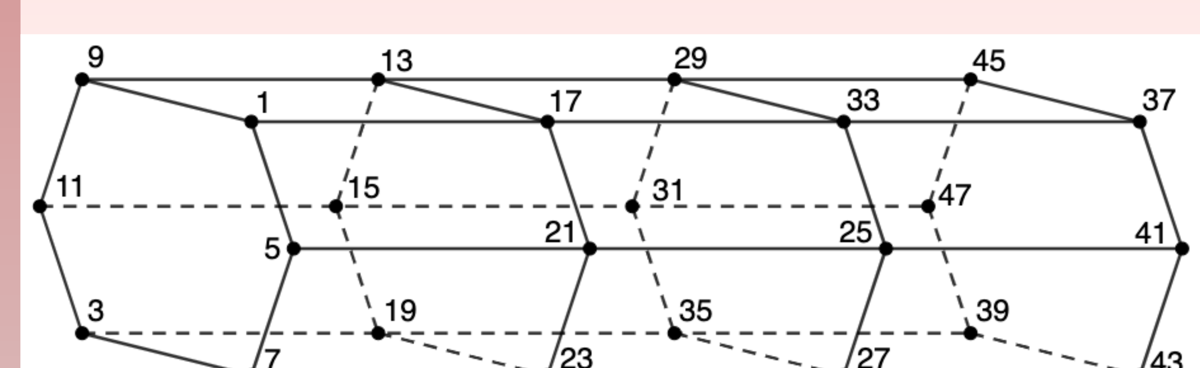
An odd prime labeling of the $C_5 \cup C_8 \cup C_3 \cup C_4$

Stacked Prism Graphs

- **Theorem 2**: All stacked hexagon graphs are odd prime.

To label any vertex $n_{i,j}$: If $i \leq 3$ see the table below. If $i > 3$, then i is of the form $i = 3a + b$, for some integers a and b so $\ell(n_{i,j}) = \ell(n_{b,j} + 36a)$.

$n_{i,1}$	$n_{i,2}$	$n_{i,3}$	$n_{i,4}$	$n_{i,5}$	$n_{i,6}$
1	9	11	3	7	5
17	13	15	19	23	21
33	29	31	35	27	25



An odd prime labeling of $Y_{6,4}$

Other combinations of cycles shown to be Odd Prime

- $Y_{3,n}$
- $Y_{5,n}$
- $Y_{2^{\ell},n}$
- Book graphs
- Möbius Ladders

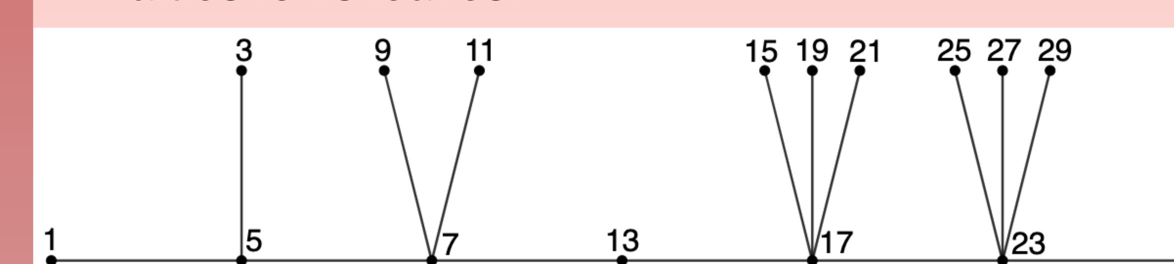
Trees

Caterpillar Graphs at most degree 5

- **Theorem 3**: All caterpillar graphs with maximum degree at most 5 are odd prime.

Begin by labeling the first vertex on the spine with 1, for the rest of the labels let m be the lowest unused odd integer. For each vertex on the spine, we label them in the following manner:

- If $\deg(v_i)$, assign $\ell(v_i) = m$.
- If $\deg(v_i) = 3$, at least one of m or $m+2$ is not a multiple of 3, assign the smallest such value as $\ell(v_i)$ and the other one to its leaf.
- If $\deg(v_i) = 4$, at most 1 of m, $m+2$, and $m+4$ is a multiple of 3, and likewise for a multiple of 5, so at least 1 of m, $m+2$, and $m+4$ is not a multiple of 3 or 5. Assign the smallest such value to v_i and the other 2 values to its leaves.
- If $\deg(v_i) = 5$, at most 2 of m, $m+2$, $m+4$, and $m+6$ is a multiple of 3, and at most 1 is a multiple of 5, so at least 1 is not a multiple of 3 or 5. Assign the smallest such value to v_i and the remaining values to its leaves.

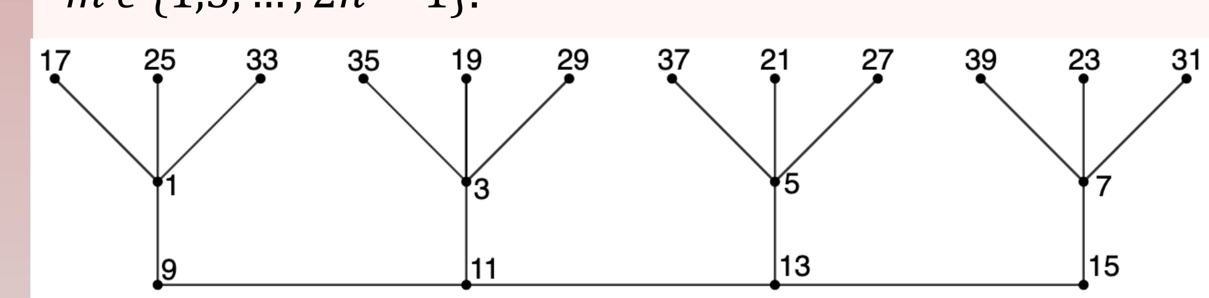


An odd prime labeling of caterpillar at most degree 5

Firecrackers

- **Theorem 4**: The firecracker graph $F_{n,k}$ has an odd prime labeling for all $n > 1$ and $k \geq 3$.

We start by labeling the n vertices on our path with the odd numbers from $2n+1$ to $4n-1$. Then by Theorem 3 in Robertson & Small's paper, there is a function h pairing the set of labels of the interior vertices $\{1,3,\dots,2n-1\}$ with the set $\{2nj+1, 2nj+3, \dots, 2n(j+1)-1\}$ such that $\gcd(m, h(m)) = 1$ for all $m \in \{1,3,\dots,2n-1\}$.



An odd prime labeling of $F_{4,5}$

Other trees shown to be Odd Prime

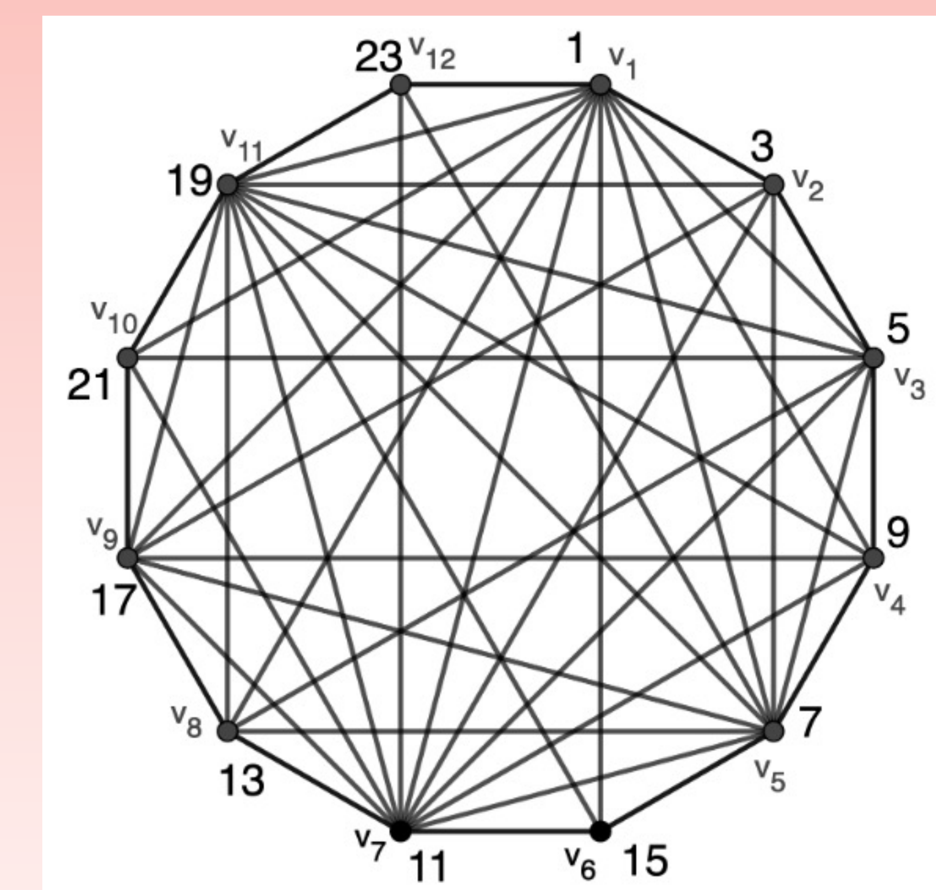
- T-toed caterpillars
- Complete binary trees

Prime Vs. Odd Prime

No examples of graphs have been found to be prime, but not have an odd prime labeling. Our expectation is that no such graphs exist, as summarized for the following conjecture

- **Conjecture 1**: Every prime graph is an odd prime.

One avenue to proving this involves the graph R_n defined in [4] as the graph consisting of vertices $v_1, v_2, \dots, v_n$ where $v_i v_j$ is an edge if and only if $\gcd(i,j) = 1$. Since all prime graphs of order n are isomorphic to a spanning subgraph of R_n. This conjecture would be proven if one could develop an odd prime labeling of R_n for any n. Namely we would need a function $\ell : \{v_1, v_2, \dots, v_n\} \to \{1, 3, \dots, 2n-1\}$ for any $n \geq 1$ such that for all $i, j \in \{1, 2, \dots, n\}$, if $\gcd(i,j) = 1$, then $\gcd\big(\ell(v_i), \ell(v_j)\big) = 1$.



An odd prime labeling of R_{12}

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