Periodic Billiard Orbits on Surfaces of Revolution

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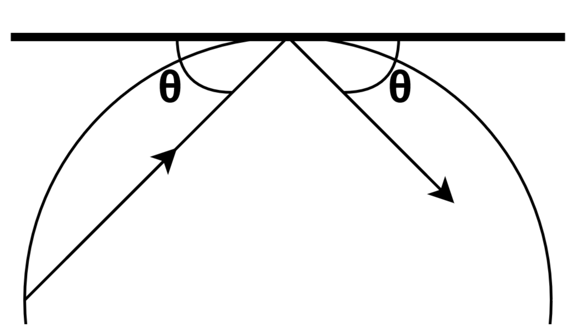
Background

Billiards

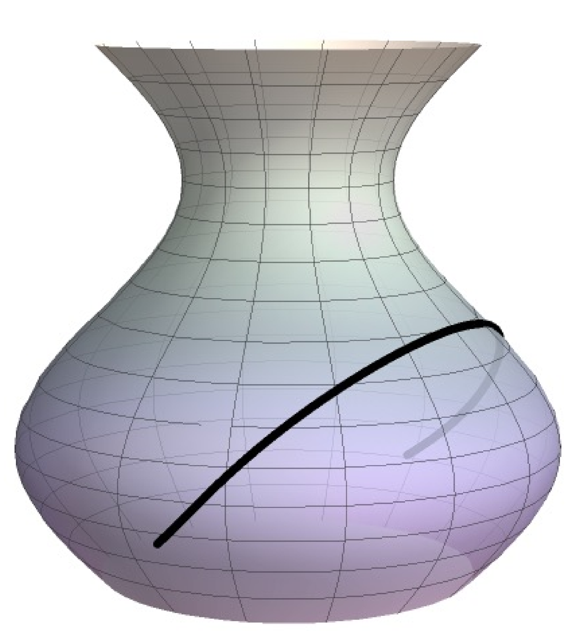
- A **billiard table** B is is a closed region whose boundary is an oriented piecewise C^2 curve
- Tables naturally admit a continuous dynamical system inspired by the game of billiards
- Particles move in straight lines and collide with the billiard boundary so that their angle of incidence is equal to the angle of reflection
- The **orbit** of a particle is its path following the billiard rules, beginning from some initial position and angle



Angle of incidence equals angle of reflection with respect to the line tangent to the billiard table

Surfaces of Revolution

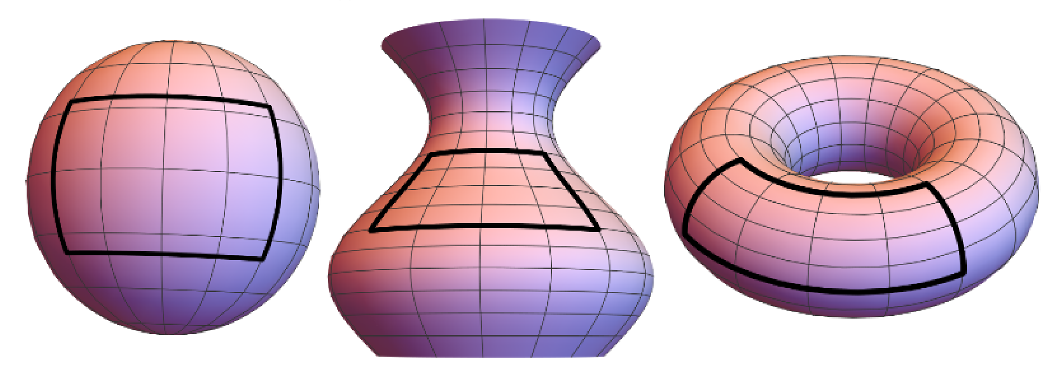
- We study billiard tables placed on **surfaces of revolution**, surfaces created by rotating C^∞ curves in the xz-plane around the $z-$axis.
- On curved surfaces, orbits travel alongs paths known as **geodesics**, which generalize the notion of a constant-velocity straight line path
- Surfaces of revolution have **parallels** and **meridians**, mirroring lines of latitude and longitude



A geodesic on a vase

Rectangular Tables

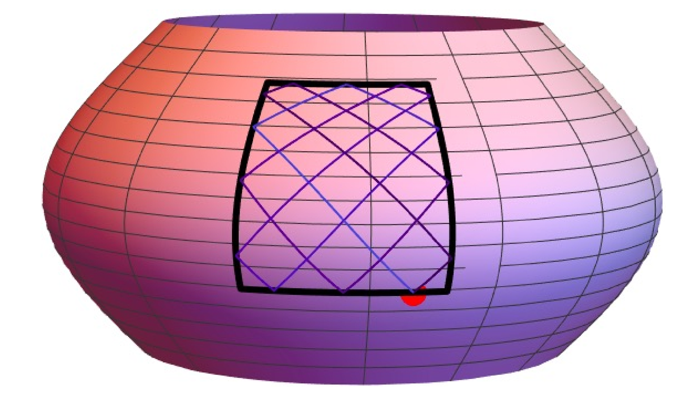
Definition 1: A $w \times h$ *rectangle* is a billiard board on a surface of revolution defined by segments of parallels on the top and bottom of coordinate width w and segments of meridians on the left and right of coordinate length h.



Rectangles on a sphere, a vase, and a torus

Periodicity

Definition 2: A billiard orbit is *periodic* if it returns to the initial point with the inital angle after n bounces. We call n the *period* of the orbit.



An example of a periodic-14 orbit on a $w \times h$ rectangle.

Rectangle Periodicity Theorem

In a $w \times h$ rectangular billiard board, an orbit γ that collides with the base of the board is periodic if and only if there exist natural numbers $i, j \in \mathbb{N}$ such that

$$i\rho_\gamma = 2jw.$$

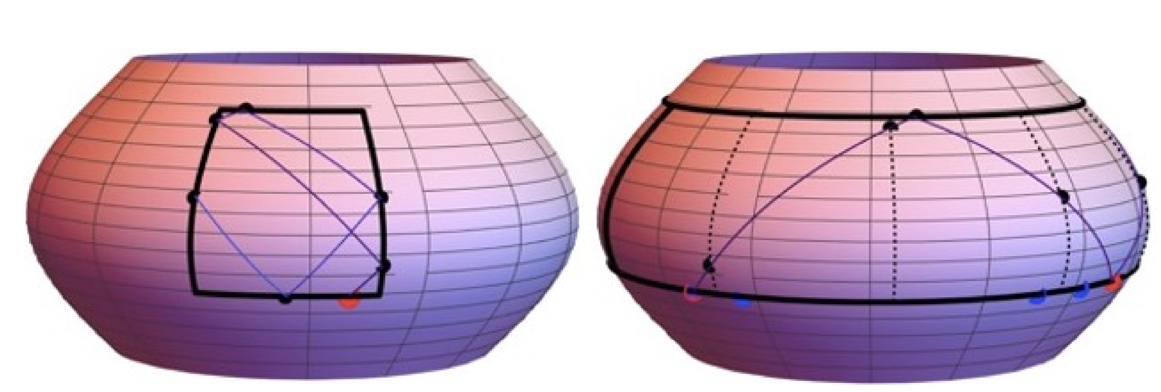
Then the period of γ is given by

$$Per(\gamma) = \begin{cases} 2i + 2j & \textit{if } \gamma \textit{ reaches the top of the rectangle,} \\ i + 2j & \textit{otherwise.} \end{cases}$$

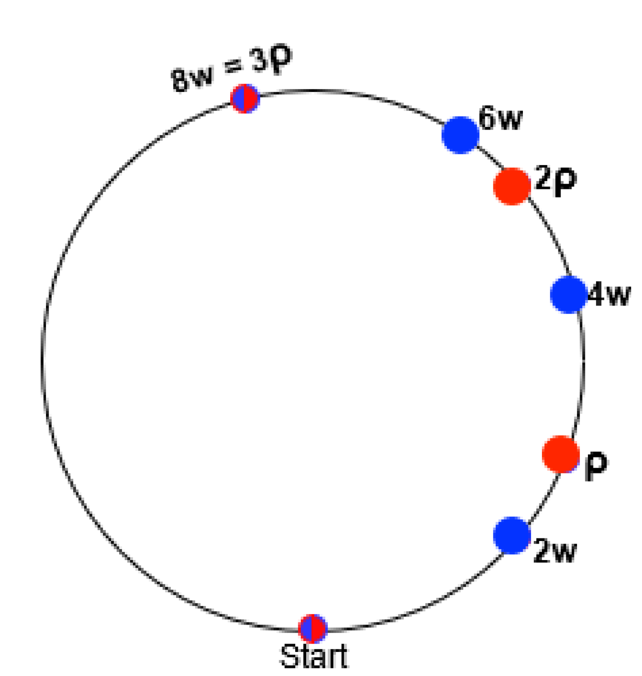
where i and j are the smallest values that satisfy the condition above.

Unfoldings and ρ_γ

A billiard orbit on a rectangle can be unfolded over meridian sides to more clearly visualize its geometric structure. With unfolding, we can quantify how rapidly an orbit γ rotates around the z-axis using a *rotation number* ρ_γ.



Unfolding



A top down view showing where ρ and w lie on an unfolded billiard table for a 14-periodic orbit

Using unfoldings, we see that periodicity occurs when ζ, the unfolding of γ, intersects an even reflection of ρ. That is, when their rotation around the z-axis are equal.

Corollaries

Corollary: An equatorial triangle on a sphere is an $h \times w$ rectangle with base at the equator and top at the north pole. All orbits on such triangles are periodic if

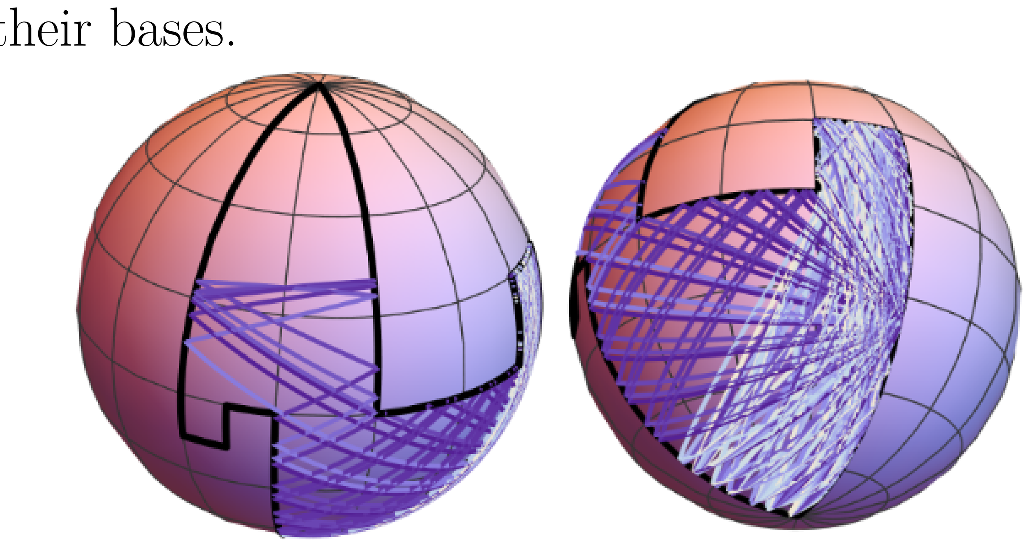
$$w = \frac{i}{2j}\pi.$$

Otherwise, no orbits are periodic.

Proof: On a sphere, all billiard orbits have $\rho = \pi$. Inserting this into our formula shows that every orbit γ on such a table is periodic with per.$(\gamma) = i + 2j$.

Corollary: There exists a family of billiard tables on the sphere that are *unilluminable*: it is impossible to reach the entire room from any given starting position. If the walls of the table were mirrored, any candle placed inside it would fail to light up the whole room, no matter where it was placed.

Proof: This follows from constructing the billiard board out of two equatorial triangles with $w = \frac{1}{2j}\pi$ and two small $h \times w$ rectangles, all connected along their bases.



Two views of light rays coming from a single point P in an unilluminable room. No matter where P is positioned in the lower triangle, the square region in the upper triangle is never illuminated.

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