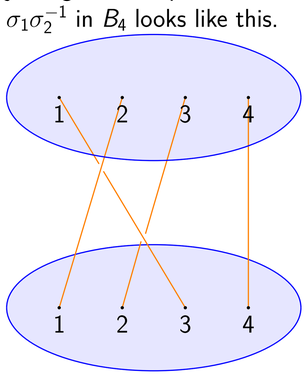
Topological Entropy of Simple Braids

Braids

For a positive integer n, the *braid group on n strands* is the group generated by $n-1$ elements $\sigma_1, \dots, \sigma_{n-1}$ subject to the relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for i with $1 \leq i \leq n-2$. A braid on n strands can be thought of as a three-dimensional object, where n points in a disc are connected to n other points in another disc below it by strands that wrap around one another. Here σ_i is a braid in which a strand going directly from position $i+1$ on top to position i on the bottom passes in front of a strand going directly from position i on the top to position $i+1$ on the bottom, while a strand goes directly from position j on the top to position j on the bottom for $j \neq i, i+1$. The multiplication operation consists of joining braids top-to-bottom. For example, the braid $\sigma_1\sigma_2^{-1}$ in B_4 looks like this.



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There is a unique homomorphism μ from B_n to the group S_n of permutations of $\{1, 2, \dots, n\}$ for which $\mu(\sigma_i)$ is the transposition that swaps i and $i+1$. (For multiplication in S_n, we compose functions left-to-right, which is different from usual conventions.) In a braid β, the strand that starts at position i on the top ends at position $(\mu(\beta))(i)$ on the bottom. The set of *simple braids* is a set of certain preimages under μ for the elements of S_n. In particular, there are $n!$ simple braids in B_n. For example, the simple braids in B_3 are ι (the identity braid), σ_1, σ_2, $\sigma_1\sigma_2$, $\sigma_2\sigma_1$, and $\sigma_1\sigma_2\sigma_1$.

Mapping class groups

The braid group B_n is isomorphic to a certain *mapping class group* of a disk. To make this group, we fix a set of n points $P_1, \dots, P_n$ inside the disk. Then we consider orientation-preserving homeomorphisms from the disk to itself that fix the boundary pointwise and permute the set of P_i's. An element of the mapping class group that we wish to consider is an equivalence class of such maps, where two maps are equivalent if there is some map yielding an isotopy between them such the map fixes the boundary pointwise and fixes each P_i. To get the isomorphism, σ_i corresponds to switching P_i and P_{i+1} by moving them in a clockwise direction relative to one another as seen from above. This relates to seeing braids as having strands: we can make the points be in a row in order from P_1 on the left to P_n on the right and let the point that starts as P_i move as the strand that starts at position i moves as it goes from top to bottom.

For such a homeomorphism, one can consider its *topological entropy*, which can be zero or positive; informally, this measures how "disordered" the map is. Then one can define the topological entropy of a braid as the infimum of the topological entropies of the maps in the equivalence class that corresponds to the braid in the mapping class group. Topological entropy of braids is of interest in areas other than just math, such as the mixing of fluids.

The Burau representation

We have a homomorphism from the braid group to $GL_{n-1}(\mathbb{Z}[t, t^{-1}])$. This is the *reduced Burau representation*. Specifically, it sends σ_i to the matrix whose (j,k) entry is t if $(j,k) = (i, i-1)$, $-t$ if $(j,k) = (i,i)$, 1 if $(j,k) = (i, i+1)$, and δ_{jk} otherwise. For example, for the reduced Burau representation of B_5, we have $\sigma_3 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t & -t & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Call this map ρ. We have from known results that the eigenvalues of the image of a braid under ρ for t on the unit circle in $\mathbb{C}$ can be used to give a lower bound for the topological entropy of the braid.

The main theorems

Theorem 1. The proportion of simple braids in B_n that have positive topological entropy is $1 - o(1)$.

Theorem 2. For $n \geq 2$, suppose that $\beta \in B_n$ is a simple braid corresponding to the permutation π. Then we can explicitly determine the entries of the image of β under the Burau representation. Namely, consider arbitrary i and j with $1 \leq i, j \leq n-1$. Let u be the number of $k \in \{1, \dots, n\}$ such that $k > j$ and $\pi^{-1}(k) \leq i$. (We let π^{-1} be the inverse of π.) Let $a_{i,j}$ be the (i,j) entry (i.e., in the ith row and jth column) of the image of β under the Burau representation. Then:

- if $\pi^{-1}(j) \leq i < \pi^{-1}(j+1)$, then $a_{i,j} = t^u$;
- if $\pi^{-1}(j+1) \leq i < \pi^{-1}(j)$, then $a_{i,j} = -t^u$;
- otherwise, $a_{i,j} = 0$.

For example, for the simple braid $\sigma_4\sigma_3\sigma_2\sigma_3\sigma_4 \in B_5$, its image under the reduced Burau representation is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 0 & 0 & -t \\ t^2 & -t^2 & t & -t \\ t^3 & -t^3 & 0 & 0 \end{bmatrix}$.

Focusing on one cycle of a permutation

For a braid β, if we have a cycle $a_1 \mapsto a_2 \mapsto \dots \mapsto a_k \mapsto a_1$ in the permutation $\mu(\beta)$, then we can consider a new braid β' on k strands formed by taking only the strands starting and ending at the a_i positions.

Lemma 1. If β' has positive topological entropy, then β has positive topological entropy.

Proof. This, essentially, follows from the definitions.

Combinatorial results about cycles of permutations

The following result is already known from combinatorics:

Proposition. Let t_i be an indeterminate for all positive integers i. For every permutation π in S_n, if π has c_1 cycles of length 1, c_2 cycles of length 2, ..., c_n cycles of length n, then let $Z(\pi) = t_1^{c_1} t_2^{c_2} \dots t_n^{c_n}$. Let $\tilde{Z}(S_n) = \Sigma_{\pi \in S_n} Z(\pi)$. Let x be an indeterminate. Then

$$\Sigma_{n \geq 0} \tilde{Z}(S_n) \frac{x^n}{n!} = \exp\left(\Sigma_{i \geq 1} t_i \frac{x^i}{i}\right).$$

If we set $t_{k(2m+1)} = 0$ for all nonnegative integers m and $t_i = 1$ for all other i and do some calculations, we can obtain the following.

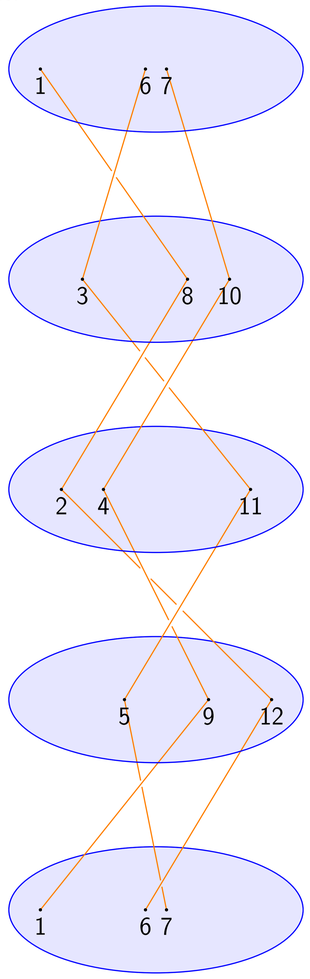
Lemma 2. Let k be a positive integer. Then the proportion of permutations of $\{1, \dots, n\}$ that have no cycle with length of the form $k(2m+1)$ for nonnegative integer m (i.e., no cycles of length $k, 3k, 5k, \dots$) is $O(n^{-1/(2k)})$.

Then the following is readily deduced by considering $k = 63 \cdot 2^{i-1}$ in Lemma 2 for $i = 0, \dots, k-1$. It will be useful later.

Lemma 3. For any positive integer r, the proportion of permutations of $\{1, \dots, n\}$ that have at least r cycles whose lengths are at least 63 and are divisible by 3 is $1 - o(1)$.

Reducing to a three-strand braid

Let m be a positive integer. Let β be a simple braid on $3m$ strands for which $\mu(\beta)$ is a $3m$-cycle. We can get a new braid γ by taking β^m and deleting all but three strands as in Lemma 1 (and relabeling positions as needed). For example, if β is the twelve-strand simple braid corresponding to the cycle $1 \mapsto 8 \mapsto 2 \mapsto 12 \mapsto 6 \mapsto 3 \mapsto 11 \mapsto 5 \mapsto 7 \mapsto 10 \mapsto 4 \mapsto 9 \mapsto 1$, then γ looks like this.



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If the new braid γ has positive topological entropy, then so does β.

We can define P_m to be the set of m-tuples $(\beta_1, \dots, \beta_m)$ of simple braids on three strands such that the product $\beta_1 \cdot \dots \cdot \beta_m$ maps to a three-cycle under μ. This set P_m has $2 \cdot 6^{m-1}$ elements. Then the operation described above yields an element of P_m for each of the $(3m-1)!$ possibilities for β, by taking the three-strand braid at each "level"; furthermore, we have $\gamma = \beta_1 \dots \beta_m$. One can also show that each of the elements of P_m is produced by the same number of $3m$-cycles, as the "levels" are almost completely independent of one another.

Three-strand braids and the Burau representation

We now study P_m in more detail. For an element $(\beta_1, \dots, \beta_m)$ of P_m, let $\xi((\beta_1, \dots, \beta_m)) = \beta_1 \cdot \dots \cdot \beta_m$. By taking $t = -1$ in the reduced Burau representation of B_3, we get a homomorphism, which we will call f, from B_3 to $SL_2(\mathbb{Z})$, under which $\sigma_1 \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\sigma_2 \mapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

(This homomorphism is of prior interest, appearing in other works.)

Using the earlier fact about the eigenvalues of the image of a matrix under the Burau representation for certain values of t, together with the fact that $\det(f(\beta)) = 1$ for all $\beta \in B_3$, one can obtain the following result.

Lemma 4. For any $\beta \in B_3$, if $|\mathrm{tr}(f(\beta))| > 2$, then β has positive topological entropy.

We can then show the following.

Lemma 5. For any positive integer $m \geq 21$, the number of elements $(\beta_1, \dots, \beta_m)$ to P_m that map to a braid of positive topological entropy under ξ is at least $2 \cdot 6^{m-21}$.

To prove this it suffices to show that whatever we choose for the first $m-21$ coordinates, we have at least two ways to choose the last 21 coordinates to yield a positive-topological-entropy braid. In fact, for each of two suitable choices for β_{m-20}, at least one of the following choices for $(\beta_{m-19}, \dots, \beta_m)$ works:

- (ι, ι),
- $(\sigma_1, \sigma_1, \sigma_1, \sigma_1, \sigma_2, \sigma_2, \sigma_2, \sigma_2, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota)$,
- $(\sigma_2, \sigma_2, \sigma_2, \sigma_2, \sigma_1, \sigma_1, \sigma_1, \sigma_1, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota, \iota)$, or
- $(\sigma_1, \sigma_1, \sigma_1, \sigma_1, \sigma_1, \sigma_1, \sigma_2, \sigma_2, \sigma_2, \sigma_2, \sigma_2, \sigma_2, \sigma_2, \sigma_1, \sigma_1, \sigma_1, \sigma_1, \sigma_2, \sigma_2)$.

One can show this using a bit of number theory and casework, using the fact that $\det(f(\beta)) = 1$ for all $\beta \in B_3$.

Finishing the proof of Theorem 1

For any $\varepsilon > 0$ (assume WLOG that $\varepsilon \leq 1$), choose a positive integer r that is large enough so that $(1 - \frac{\varepsilon}{2})(1 - (1 - \frac{1}{6^{20}})^r) > 1 - \varepsilon$, and choose N sufficiently large such that, for all $n \geq N$, we have that, if we consider the proportion of permutations in S_n that have at least r good cycles, then this proportion is greater than $1 - \frac{\varepsilon}{2}$. Then, for all positive integers $n \geq N$, the proportion of simple braids in B_n that have positive topological entropy is greater than $1 - \varepsilon$. Informally, this is because almost all permutations in S_n have many long cycles (i.e., cycles with length at least 63) whose length is divisible by 3, by Lemma 3, and each such cycle has a probability of at most $1 - \frac{1}{6^{20}}$ of yielding a topological-entropy-zero braid, by other earlier results, so we get the desired; Theorem 1 is shown.

Proof of Theorem 2

Given an arbitrary simple braid in B_n, it is known that it can be written as $(\sigma_{i_1} \cdot \dots \cdot \sigma_2 \cdot \sigma_1) \cdot (\sigma_{i_2} \cdot \dots \cdot \sigma_2) \cdot \dots \cdot (\sigma_{i_{n-1}} \cdot \dots \cdot \sigma_{n-1})$ (where some of the products in parentheses might be empty). Using this, one can show Theorem 2 by induction; careful computation is involved.

Further directions

It may be interesting to try to improve Theorem 1 to get better upper bounds on the number of simple braids in B_n with topological entropy zero, or even an exact formula. It is possible that Theorem 2 could help with this, by considering the eigenvalues of the matrices from Theorem 2. It may also be interesting to consider lower bounds for the number of simple braids in B_n with topological entropy zero. It also may be interesting to consider generalizations of the set of simple braids.