# Bootstrapping the Likelihood Ratio Test for Change Point Analysis

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## Introduction

Background: The basis of change point analysis is to detect changes in distribution (behavior) from a sequence of observations because the distribution may change as the sequence progresses [1]. For example, suppose that we are in charge of monitoring some process to ensure that its observations stay within some range for quality control. That way, we may be able to minimize loss should we detect changes in the behavior of these observations. To improve the robustness of the change-point analysis, we present a simulation study which applies the nonparametric bootstrap to the likelihood ratio (LR) test. The results show that the bootstrapped LR test is robust for non-normal observations.

Set-up: Using the binary segmentation method, the change point problem under the assumption that only one change point exists is equivalent to a hypothesis test with null hypothesis,

$$H_0\colon \mu_1 = \mu_2 = \cdots = \mu_n,$$

and alternative hypothesis,

$$H_a\colon \mu_1 = \mu_2 = \cdots = \mu_k \neq \mu_{k+1} = \cdots = \mu_n,$$

for some $1 < k < n$, where $k$ is the unknown change point location.

Let $\{x_i\}_{i=1}^n$ be a sequence of independent and homoscedastic random observations. Furthermore, let

$$S = \sum_{i=1}^{n}(x_i - \bar{x})^2, \quad S_k = \sum_{i=1}^{k}(x_i - \bar{x}_1)^2 + \sum_{i=k+1}^{n}(x_i - \bar{x}_2)^2.$$

The difference $S - S_k$ is mathematically equivalent to the LR statistic assuming normality. We wish to find $k$ such that this difference is maximized, as this $k$ corresponds to the most likely location of a change point. Thus, we take our test statistic to be

$$U = \max_{1<k<n} \sqrt{S - S_k}.$$

The asymptotic distribution of $U$ is obtained after a slight transformation. Let

$$\mathcal{U} = a_n^{-1}(U - b_n),$$

where $a_n = (2 \log\log n)^{-1/2}$ and $b_n = a_n^{-1} + 2^{-1} a_n \log\log\log n$. Then, $\mathcal{U}$ asymptotically has the cumulative distribution function (cdf)

$$F_{\mathcal{U}}(x) = e^{-2\pi^{1/2} e^{-x}}, -\infty < x < \infty.$$

In practice, $F_{\mathcal{U}}(x)$ gives an approximate $p$-value, and we reject $H_0$ when $p$-value $< \alpha$ for some desired significance level $\alpha$. However, $F_{\mathcal{U}}(x)$ may not be suitable for a small sample size [2].

## Nonparametric Bootstrap Implementation

### Nonparametric Bootstrap:

The nonparametric bootstrap offers a robust alternative to the large-sample approximation or parametric bootstrap for statistical inference. It is valid for a wide range of data-generating processes.

### Algorithm for Nonparametric Bootstrap:

1. We split the original data at $k$ for which $S - S_k$ is maximized. Then, we subtract the respective sample mean from each section so that the centered data can be resampled.
2. We generate a bootstrap sample from the centered data using replacement. We denote the $b$-th resample by $\{u_{b1}, u_{b2}, \ldots, u_{bn}\}$, $b = 1, \ldots, B$, where $B$ represents a sufficiently large number of bootstrap resamples.
3. Using $B$ resamples, we compute $B$ test statistics $\{U_1^\star, U_2^\star, \ldots, U_B^\star\}$, where $U_b^\star$ is the LR test statistic for the $b$-th resample, $b = 1, 2, \ldots, B$.
4. We obtain the bootstrap $p$-value by $p = \#\{U < U_b^\star, b = 1, \ldots, B\}/B$, where $U$ is the LR test statistic from the original data.

### Simulation Set-up:

It is important to verify the validity of the nonparametric bootstrap method. The method is tested with different types of distributions ($N(0,1)$ and Exp(1)). For each of these distributions, we have sample sizes of 20, 30, 50, 100, 200, and 500. Moreover, we vary the change point location to further investigate the power of the test. The robustness is assessed by observing how close the empirical Type I error rate ($\alpha_e$) is to the nominal significance level $\alpha = 0.05$ using a Monte Carlo simulation.

1. We generate $M$ samples of size $n$ from some distribution under $H_0$ (e.g., $N(0,1)$), each of which generates an observed LR test statistic.
2. Let $\{U_1, U_2, \ldots U_M\}$ be the set of observed LR test statistics from $M$ Monte Carlo samples. The bootstrap $p$-value for $U_m$, $m = 1, 2, \ldots, M$, is given by $p_m = \#\{U_m < U_{m,b}^\star, b = 1, \ldots, B\}/B$, where $U_{m,b}^\star$ is the LR test statistic for the $b$-th resample of the $m$-th Monte Carlo sample.
3. Finally, the actual Type I error rate

$$P(\text{Reject } H_0 \mid H_0 \text{ is true})$$

is approximated by the empirical Type I error rate

$$\alpha_e = \#\{p_m < \alpha, m = 1, \ldots, M\}/M.$$

## Simulation Results

| | $n$ | 20 | 30 | 50 | 100 | 200 | 500 |
|---|---|---|---|---|---|---|---|
| Dist. | $N(0,1)$ | 0.14889 | 0.11167 | 0.09044 | 0.07000 | 0.06056 | 0.05833 |
| | Exp(1) | 0.14189 | 0.11733 | 0.09033 | 0.06889 | 0.06033 | 0.05544 |

Table 1: Empirical Type I error rates with $M = 9000$ and $B = 1000$ at $\alpha = 0.05$.

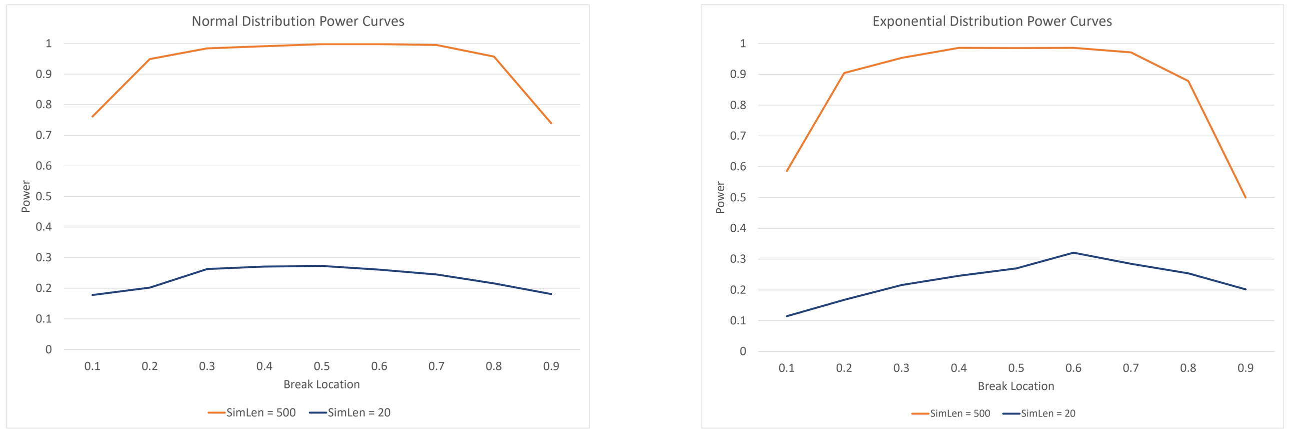


Figure 1: Power curves for standard normal and exponential distributions with $M = 1000$ and $B = 1000$ at $\alpha = 0.05$. The top orange curves are for $n = 500$ and the bottom blue curves are for $n = 20$. The change location expressed in terms of the proportion is on the $x$-axis. Each change point has a jump size of $0.5$.

### Observations:

- The empirical Type I error rate converges to the nominal significance level at $\alpha = 0.05$, but a moderate sample size $n \geq 100$ is required to obtain a robust test.
- The power curves show an increase in the power of the test as the sample size increases. Moreover, the power curves achieve their maxima at around $0.5$, indicating that the LR test statistic can most easily detect a mean change if that occurs at around the midpoint of the data.

## Conclusions

The nonparametric bootstrap method provides a robust and powerful alternative to the asymptotic method for the detection of change point locations. With at least a moderate sample size of $n = 100$ or larger, the nonparametric bootstrap method shows promising simulation results, even if the original data are non-normal.

## References

[1] J. Chen, A. Gupta, and J. Pan, "Information criterion and change point problem for regular models," *Sankhyā: The Indian Journal of Statistics*, vol. 68, no. 2, pp. 252–282, 2006.

[2] N. Prime, "Likelihood Ratio Approach for Detecting Mean Changes." Senior Project Report, Western Washington University, 2020.