On the Distance Spectra of Extended Double Stars

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Introduction

The distance matrix of a connected graph is defined as the matrix in which the entries are the pairwise distances between vertices. The distance spectrum of a graph is the set of eigenvalues of its distance matrix. A graph is said to be determined by its distance spectrum if there does not exist a non-isomorphic graph with the same spectrum. The question of which graphs are determined by their spectrum has been raised in the past, but it remains largely unresolved. We prove that extended double stars are determined by their distance spectra.

An *extended double star*, denoted by $T(a, b)$, is the graph formed by joining the centers of $K_{1,a}$ and $K_{1,b}$ to a common vertex.

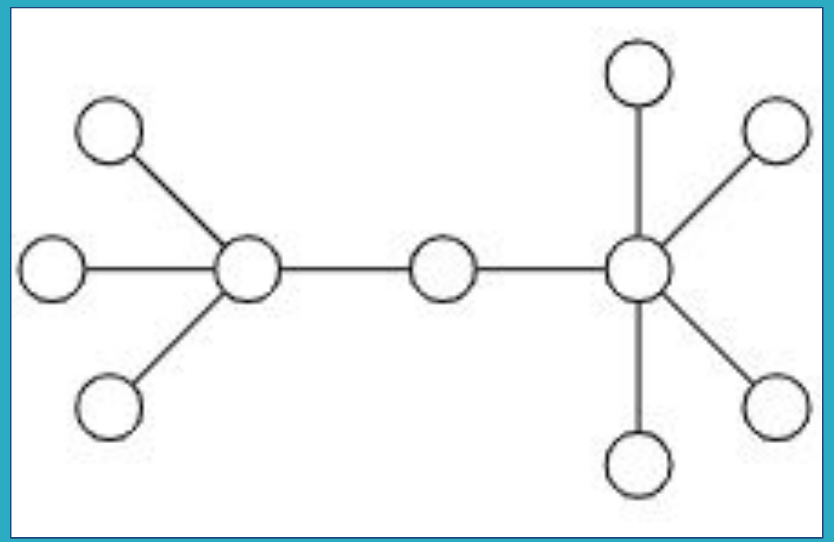

Fig 1. $T(3, 4)$.

Previous Work and Our Results

The question of which graphs are determined by their distance spectra has been raised in [6], though this question is largely open.

Previous work in [8] has proven that the double star $S(a, b)$ is determined by its distance spectrum which provided our inspiration. Other work in [4] includes proving that K_n and the complete split graph are determined by their distance spectra. In addition, in [3] and [5], complete k-partite graphs, graphs with exactly two distance eigenvalues not equal to -1 or -3, and the friendship graph have been proven to be determined by their distance spectra. Graphs of diameter 2 follow the property that $D = 2J - 2I - A$, so the distance matrix is a generalized adjacency matrix, which is studied in [7].

We investigate a family of graphs of diameter 4, $T(a, b)$, and proves it is determined by its distance spectrum. Our proof is split into cases based on diameter and show that no non-isomorphic graphs have the same distance spectrum as $T(a, b)$.

Methodology

Theorem 1. *Let G be a graph with n vertices and distance matrix $D(G)$. Denote the n eigenvalues as $\lambda_1(D(G)), \lambda_2(D(G)), \ldots, \lambda_n(D(G))$ where $\lambda_1(D(G)) \geq \lambda_2(D(G)) \geq \cdots \geq \lambda_n(D(G))$. Let H be an induced subgraph of G with m vertices with distance spectrum $\mu_1(D(H)) \geq \mu_2(D(H)) \cdots \geq \mu_m(D(H))$. If $D(H)$ is a principal submatrix of $D(G)$, then $\lambda_{n-m+i}(D(G)) \leq \mu_i(D(H)) \leq \lambda_i(D(G))$ for $i = 1, 2, \ldots, m$.*

Lemma 1. *Let $G = T(a, b)$. Then G has distance characteristic polynomial*

$$(-\lambda - 2)^{a+b-2} p_{a,b}(\lambda),$$

where $p_{a,b}(\lambda) = 16 + 8a + 8b + 40\lambda + 36a\lambda + 36b\lambda + 8ab\lambda + 28\lambda^2 + 44a\lambda^2 + 44b\lambda^2 + 24ab\lambda^2 + 2\lambda^3 + 18a\lambda^3 + 18b\lambda^3 + 12ab\lambda^3 - 4\lambda^4 + 2a\lambda^4 + 2b\lambda^4 - \lambda^5$. When a and b are clear from context we simply use $p(\lambda) = p_{a,b}(\lambda)$.

In particular, we find that $T(1, 1)$ has distance spectrum $\{8.2882, -0.5578, -0.7639, -1.7304, -5.2361\}$. Then for $a, b \geq 1$, $D(T(1, 1))$ is a principal submatrix of $D(T(a, b))$, so we apply Theorem 1. Next, denote $c = \max(a, b)$, and use Lemma 1 to find the characteristic polynomial for $T(c, c)$. Clearly, $D(T(a, b))$ is a principal submatrix of $D(T(c, c))$, so we apply Theorem 1. The complete spectrum for $T(a, b)$ is below:

λ_1	λ_2	λ_3	λ_4	λ_5	$\cdots$	λ_{n-1}	λ_n
$[8.2882, \infty)$	$[-0.5578, 0)$	$[-0.7639, -0.4226)$	$[-1.7304, -1.5774)$	-2	$\cdots$	-2	$(-\infty, -5.2361]$

Table 1. Complete spectrum of $T(a, b)$.

Let G be a connected graph cospectral to $T(a, b)$. By Theorem 1 and the table for the spectrum of $T(a, b)$, we know P_6 is a forbidden subgraph of G, so $d(G) \leq 4$. A result from [8] yields $d(G) \neq 2$, so we conclude $d(G) = 3$ or 4.

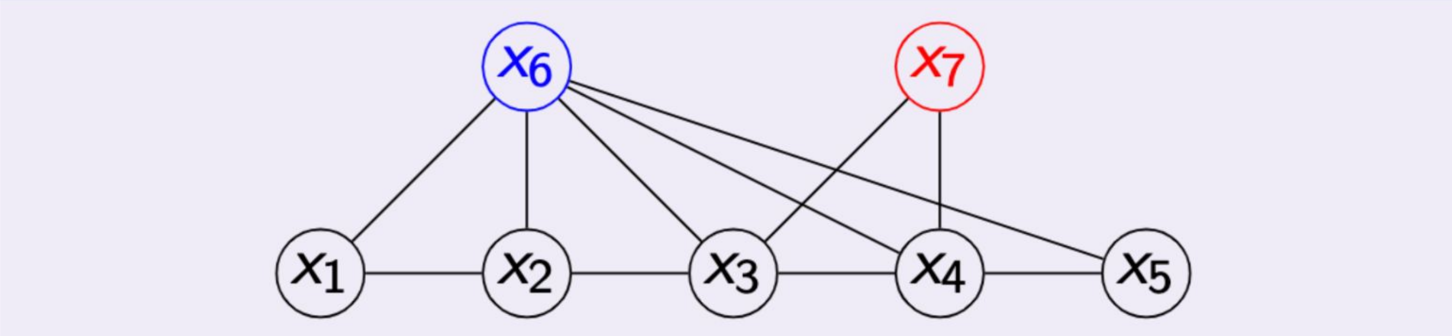

Fig 2. If $d(G) = 4$, denote $X = \{x_1, x_2, x_3, x_4, x_5\}$ the vertex set of a length 4 path. Denote by $V_i (i = 0, 1, 2, 3, 4, 5)$ the vertex subset of $V \backslash X$ consisting of vertices adjacent to i vertices of X. For $d(G) = 3$, we define $V_i (i = 0, 1, 2, 3, 4)$ analogously. Here, $x_6 \in V_5, x_7 \in V_2$.

Diameter 3

In V_2, interlacing eliminates all subgraphs except the one shown below.

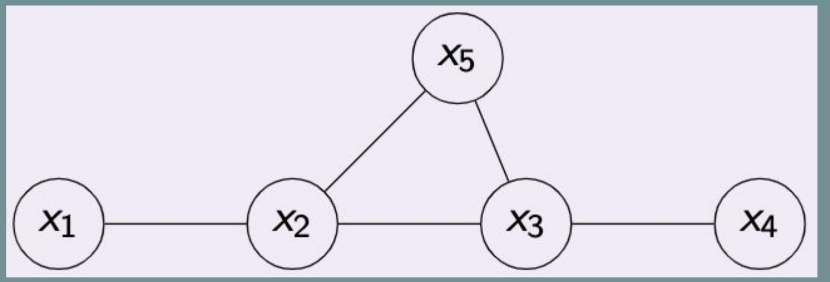

Fig 3. Hat-shaped subgraph.

Diameter 4

To prove V_5 empty, just show this subgraph is forbidden.

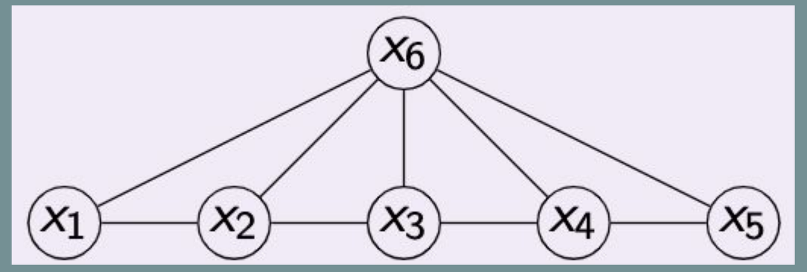

Fig 4. Subgraph with $x_6 \in V_5$.

Results

For diameter three graphs, we compute characteristic polynomials of the remaining subgraphs and compare to the result of Lemma 1 to conclude no graphs of diameter 3 are cospectral to $T(a, b)$. For diameter four graphs, we eliminate enough subgraphs to conclude any cospectral graph is isomorphic to $T(a, b)$.

Theorem. *The graph $T(a, b)$ is determined by its distance spectrum.*

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