Subgraphs of Unit Distance Graphs

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Introduction and Main Results

Introduction: The **Erdős Unit Distance Problem**, which asks for the maximum number of times that the unit distance can occur among n points (in the plane or $\mathbb{R}^3$), is one of the most well-known problems in Discrete Geometry. [5] [2] Many variants have been studied in the work of Elekes, Pach, and others. [1] Our work is on bounds on the **maximum number of isomorphic copies of a given subgraph in a unit distance graph**, the graph whose edges represent unit distances between the n points. These kinds of questions were studied in [4] and in [3], and we continue the work of these papers by solving the following three problems: (throughout the whole project, k is assumed to be constant with respect to n, and we look for sharp asymptotic bounds in n):

1. Among n points on a sphere of radius $\frac{1}{\sqrt{2}}$, how many paths or cycles of length k ($P_k^S(n)$ and $C_k^S(n)$, respectively) can be found such that the distance between consecutive points is one?
2. Among n points in $\mathbb{R}^3$, how many 3-regular graphs on k vertices can be found such that the distance between v_i, v_j is unit if v_iv_j is an edge in the graph?

Our Main Results:

1. For any fixed $k \geq 1$, the **number of unit distance paths** on k vertices on the sphere is
$$P_k^S(n) = \tilde{\Theta}\left(n^{\lfloor 2(k+3)/5 \rfloor}\right), \text{if } k = 0,1,3,4 \pmod 5, \tilde{\Theta}\left(n^{\lfloor 2(k+3)/5 \rfloor - 2/3}\right), \text{if } k = 2 \pmod 5.$$
2. Theorem 2: $C_k^S(n)$ is the **number of unit distance cycles** on k vertices on the sphere. We have $C_4^S(n) = \Theta(n^2)$, and for any $k \geq 5$, with the exception of $k = 6,7,9$ we have We have $C_4^S(n) = \Theta(n^2)$, and for any $k \geq 5$, with the exception of $k = 6,7,9$ we have
$$C_k^S(n) = \tilde{\Theta}\left(n^{\lfloor 2k/5 \rfloor}\right), \text{if } k = 0,1,3,4 (\text{mod } 5), \tilde{\Theta}\left(n^{\lfloor 2k/5 \rfloor + 1/3}\right), \text{if } k = 2 (\text{mod } 5).$$
3. Theorem 3: Let G be a fixed 3-**regular graph** on k vertices. The maximum number of unit distance subragraphs isomorphic to G determined by a set of n points in $\mathbb{R}^3$ is
$$\tilde{O}\left(n^{k/2}\right)$$

Example Construction

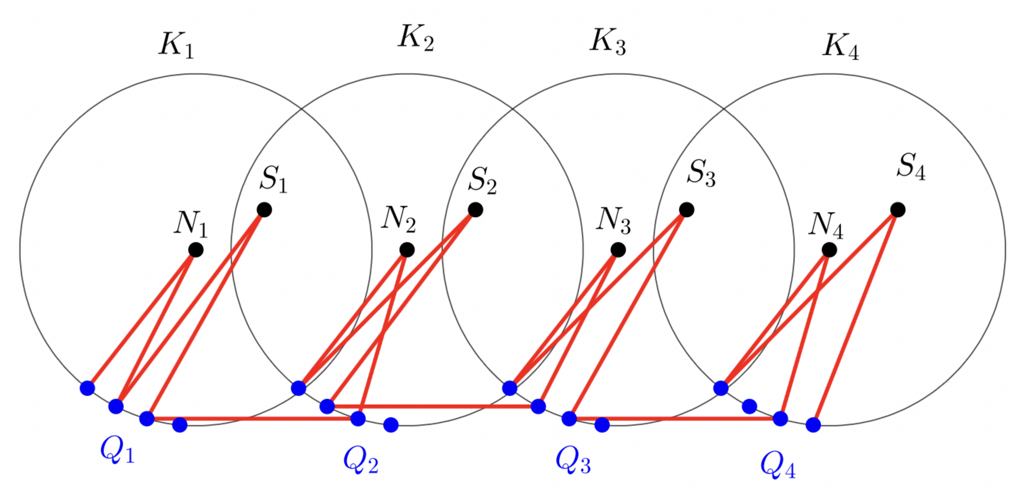

Figure 1

Explanation of Constructions

Figure 1: A construction giving the best possible bound for the number of paths of length k (in the 2 mod 5 case, this construction can be modified slightly to take advantage of previous Unit Distance constructions). The K_is represent great circles and the S_is represent poles of the great circles. We place one of our n points at each S_i and $\Omega(n)$ of our points on each K_i. This construction can be slightly modified to give the optimal construction for cycles.

In $\mathbb{R}^3$, we can give a similar construction which takes advantage of the third dimension; this leads to a construction which gives the upper bound for Theorem 3.

First Main Method for Obtaining Upper Bounds

To achieve bounds for paths and cycles on the sphere, our main idea was to extract as much information about the **unit distance graph** (the graph whose vertices are the n points and whose edges represent unit distances), and then use graph theory and combinatorics in and of itself to achieve upper bounds.
An easy case that can be quickly explained is when $k = 3$; for larger cases, one uses induction and some information given by geometric results.

Second Main Method for Obtaining Upper Bounds

To achieve bounds for the 3-regular graph problem, we used ideas from a paper by Agarwal and Sharir. Specifically, we applied the **Cutting Lemma**, a precursor to the more general polynomial methods which have recently been used to make progress on the equally famous Distinct Distances Problem. We apply the Cutting Lemma one time for each vertex, describing an **algorithm that assigns weights** to each vertex in our graph so that the Cutting Lemma gives us an optimal upper bound.

Concluding Remarks

Many questions remain along this line of research. Here are a few questions we still wonder about:

1. The work of Frankl and Kupavskii solves the question of how many unit distance paths of length k can be found among n points in $\mathbb{R}^2$ in infinitely many cases, except for those which are provably as difficult as the Unit Distance Problem itself, and does similarly in $\mathbb{R}^3$. However, very little has been proved on the number of **unit k-cycles in $\mathbb{R}^2$ or** $\mathbb{R}^3$, although we solved the problem in all but a few small cases on the sphere. Even in small cases, the problem is still open.
2. Specifically, the case when $k = 4$ in $\mathbb{R}^2$ is still unknown. A lower bound is $u_2(n)$ (the answer to the Unit Distance Problem), and an upper bound is $\tilde{O}(n^{\frac{5}{3}})$.
3. The work of Frankl and Kupavskii mentions the more general question of **unit distance trees** - finding bounds for the number of subgraphs of the unit distance graph isomorphic to a given tree. In R^2, this seems like a very difficult problem. On the sphere, it seems more approachable; we were able to show some preliminary results for **sparse trees** (where sparse means here that the minimum path connecting any two degree three or higher vertices is at least some constant length). Can these results be improved to a less restricted set of trees?
4. The construction above for 3-regular graphs gives an optimal construction for bipartite 3-regular graphs. Is the lower bound also true for non-bipartite 3-regular graphs? Even for small, fixed graphs (such as the Petersen graph), we do not have complete answers.

References

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