Simon's Conjecture for 1-decomposable Complexes

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Introduction

A Simplicial Complex Δ on a ground set $[n]$ is a collection of sets which are closed under inclusion. If $\tau \in \Delta$ and $\sigma \subseteq \tau$, then $\sigma \in \Delta$. We call each set a face of Δ.

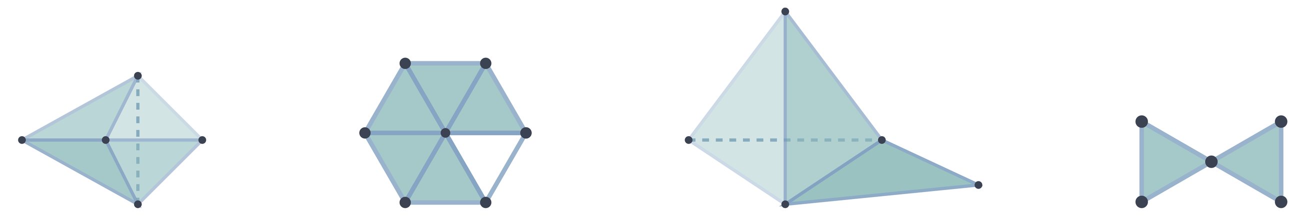


Fig. 1: Some examples of simplicial complexes

- The k-skeleton of a simplicial complex is the set of faces that are at most k-dimensional.
- A face is a facet if it is not contained in any larger face.
- A simplicial complex is flag if every clique of edges is a face.
- A shelling move occurs when a d-dimensional facet is added to a simplicial complex such that it intersects the complex in only $(d-1)$-dimensional faces.
- A simplicial complex is shellable, if it can be built from the empty complex via shelling moves.
- A simplicial complex is stuck if it is shellable, there are missing facets and no new shelling moves exist.

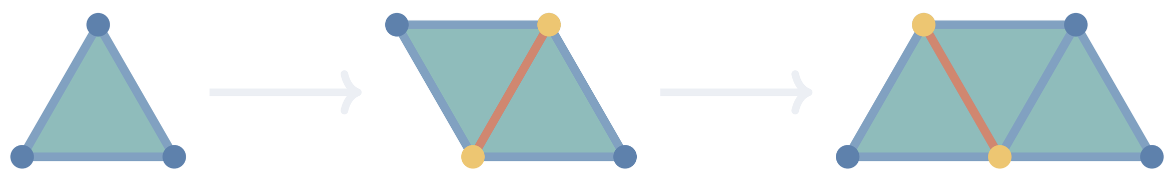


Fig. 2: A shellable simplicial complex

Simon's Conjecture
No stuck complexes exist.

Simon's conjecture is known for simplicial complexes of dimension 0, 1, 2, $n-1$, $n-2$, $n-3$.

k-Decomposable Complexes

Let Δ be the simplicial complex at the top of the tree below. For each highlighted face F, the left branch is the link of F and the right branch is the deletion of F:

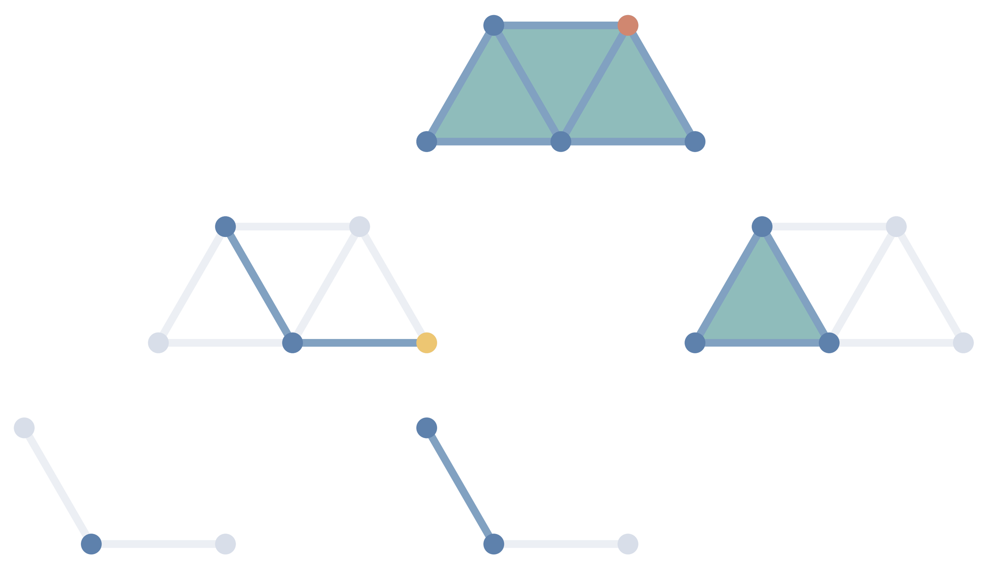


Fig. 3: A vertex decomposition of our complex

We say Δ is k-decomposable if we can decompose over $\leq k$-dimensional faces to get a collection of complexes with only one facet. Authors of [1] proved that there are no 0-decomposable stuck complexes.

Results

A new property of stuck complexes [TSU REU 2021] Suppose Δ is a stuck complex, then for all adjacent facets $A = \{a, z_1, \cdots, z_d\}$, $B = \{b, z_1, \cdots, z_d\}$ the edge $\{a, b\}$ must be contained in Δ.

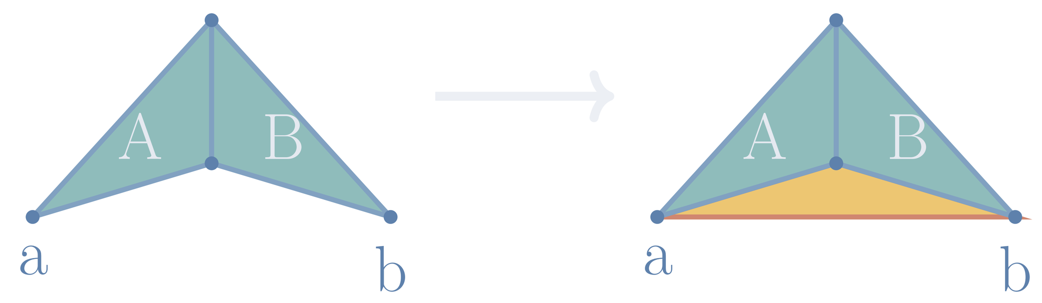
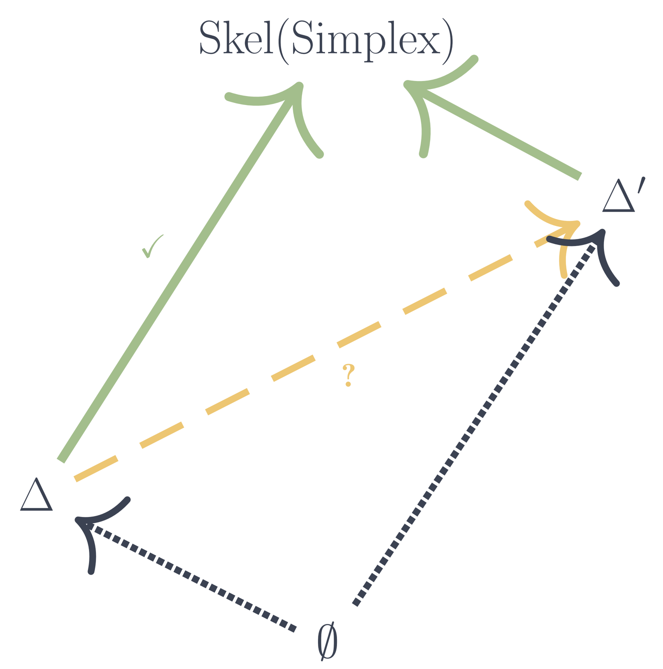


Fig. 4: Doing a shelling move using using $\{a, b\}$.

Flag 1-decomposable are not stuck [TSU REU 2021]
There are no stuck complexes which are 1-decomposable and flag.

Since there are no stuck 0-decomposable complexes, for any 0-decomposable complex, Δ, there exists an order in which we can add all of its missing facets via shelling moves. Can we find such an order for the missing facets $\Delta' - \Delta$ where Δ' is a 0-decomposable complex which contains Δ?



Counterexample [TSU REU 2021] A a 0-decomposable complex which cannot be completed to a larger 0-decomposable complex via shelling moves. This is an example used in [2] which we then checked for this property.

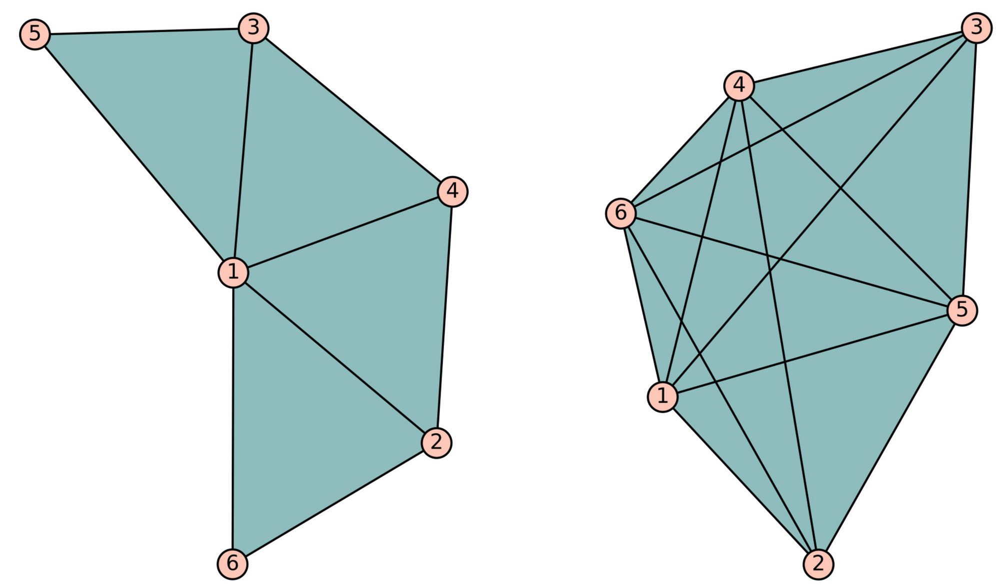


Fig. 6: A 0-decomposable complex (left) that cannot be completed to a larger complex (right).

Translating to 1-decomposable

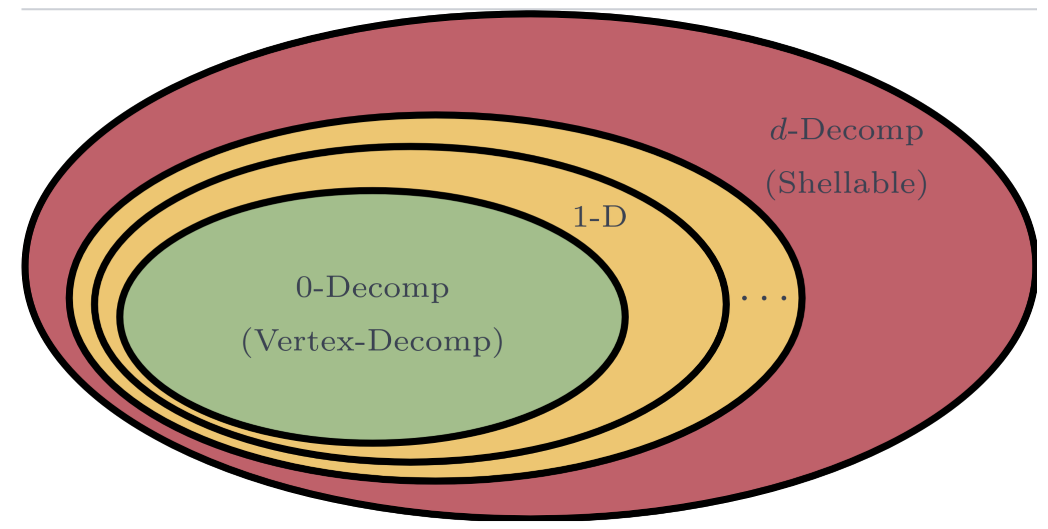


Fig. 7: A 0-decomposable complex (left) that cannot be completed to a larer complex (right).

1-Decomposable Stuck Conjecture
There are no 1-decomposable stuck complexes.

- For the proof of 0-decomposable, we add facets while keeping the same decomposing face.
- This method fails for 1-decomposable complexes by a counterexample.
- We cannot use the same approach for 1-decomposable complexes.

Further Questions

- Generally, 1-decomposability is much weaker than 0-decomposability.
- The difference is less pronounced in 3 dimensions.
- In 3 dimensions, the link of an edge is a graph.

3-Dimensional Stuck Conjecture There are no 3-dimensional, 1-decomposable stuck complexes.

Acknowledgments

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References

[1] Michaela Coleman et al. "Completing and Extending Shellings of Vertex Decomposable Complexes". In: (2020).

[2] Sonoko Moriyama and Fumihiko Takeuchi. "Incremental construction properties in dimension twoshellability, extendable shellability and vertex decomposability". In: *Discrete Mathematics* 263.1 (2003), pp. 295–296. ISSN: 0012-365X.