Numerically solving polynomial systems using Khovanskii bases

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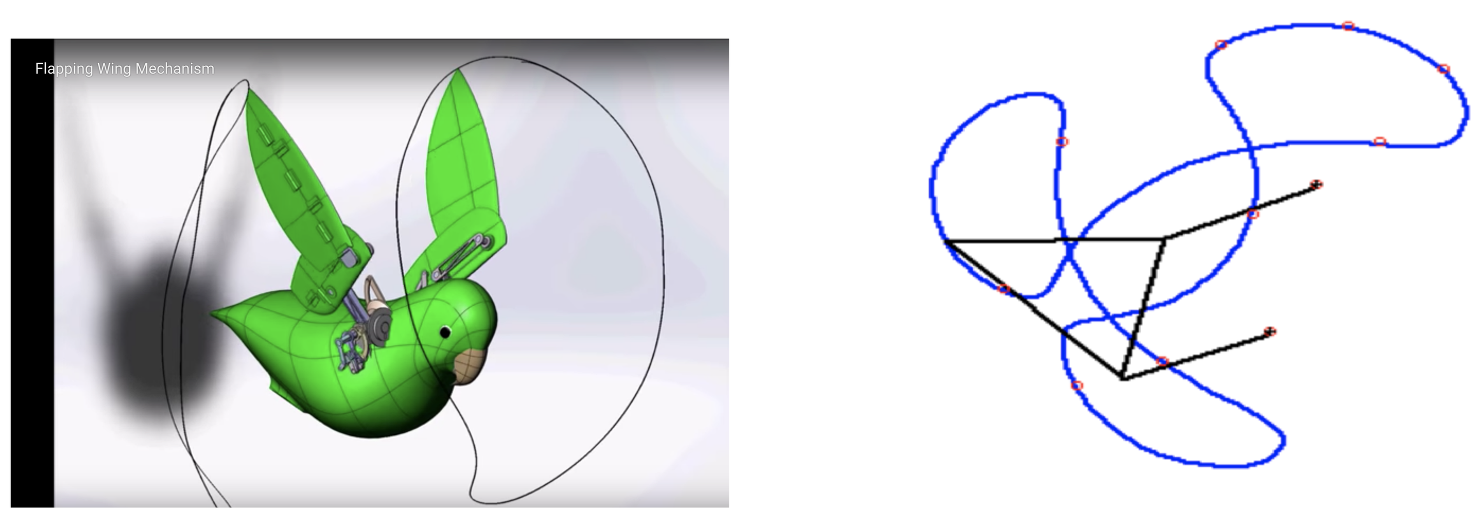
ABSTRACT

Why solve polynomial systems?

Polynomial systems occur throughout mathematics and its applications. For instance, polynomial systems arise in robotics when describing and designing mechanisms, see Figure 1.



(a) Wing mechanism, see [5] **(b)** Four-bar mechanism, see [4]

Figure 1: Some mechanism designs described by polynomials.

Polynomial systems in applications can be difficult to solve.
Our goal is to develop efficient numerical algorithms for solving polynomial systems.

Homotopies for solving systems

Homotopy continuation algorithms are useful numerical methods for quickly solving polynomial systems. Some features include:

- Solves a *target* polynomial system $F(x)$ using a similar, simpler *start system* $G(x)$ and family of polynomials $H(x;t)$.
- $H(x;t)$ defines paths interpolating between the solutions of F, denoted $\{F=0\}$, and the solutions of G, denoted $\{G=0\}$.
- Paths are tracked numerically from $t=0$ to $t=1$.
- Paths are independent: ideal for fast parallel computation.

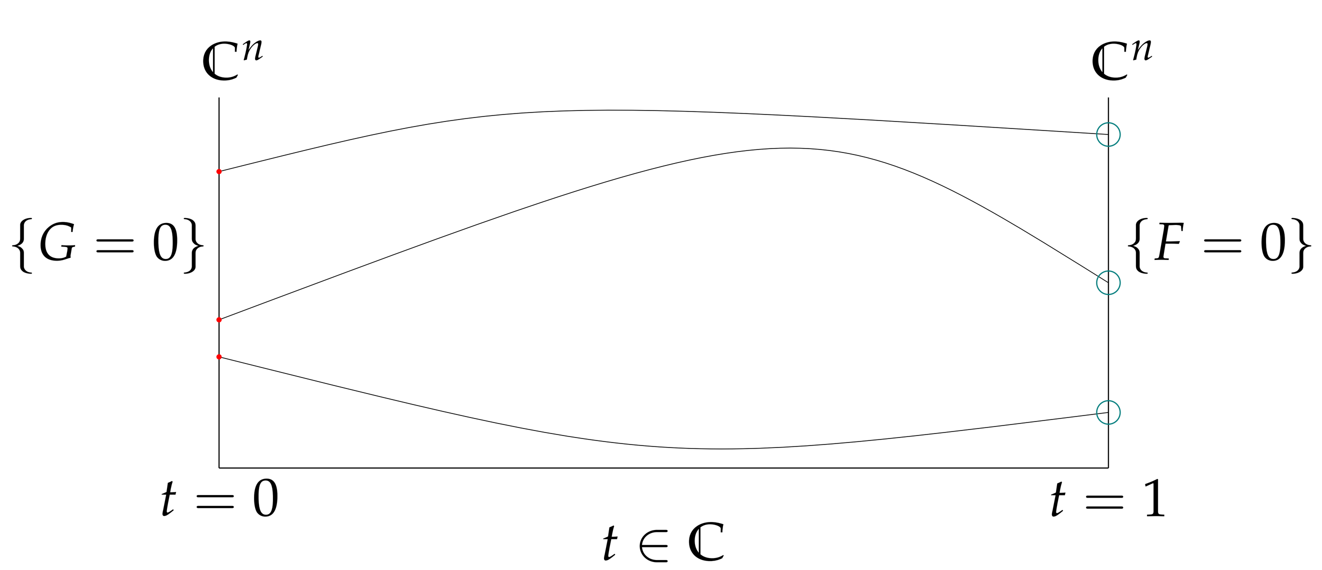


Figure 2: Cartoon of homotopy continuation for solving a system F.

Definition (Optimal homotopy)**.** A homotopy is *optimal* if $\#\{G=0\}=\#\{F=0\}$ (counted with multiplicity) and these solution sets are connected by smooth paths.

Homotopy continuation challenges

Some numerical issues may arise in these homotopy continuation algorithms. For example:

- *Paths crossing*. Avoided by the *gamma-trick*.
- *Singular solutions to F*. Handled through *endgames*.
- *Diverging paths*. Only solved by choosing a start system G such that $\#\{G=0\}=\#\{F=0\}$ (counted with multiplicity).

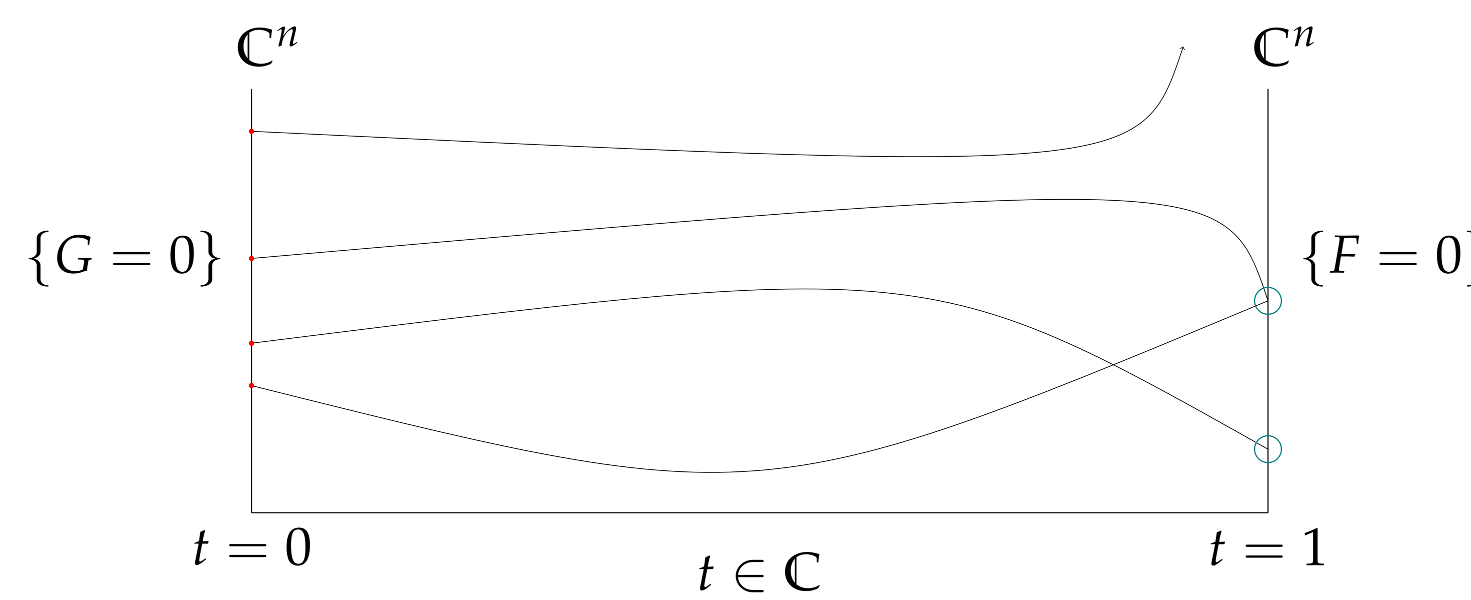


Figure 3: Path-tracking challenges in homotopy continuation.

Optimal homotopy algorithms will, in particular, avoid expensive, extraneous diverging paths.

QUESTION

Can we develop an optimal homotopy algorithm for solving polynomial systems?

Previous work

Many algorithms compute a bound d on $\#\{F=0\}$ and then generate a simple start system G with d solutions.

Homotopy	# Points in start system
Total-Degree	Bézout number
	$\geq$
Multihomogeneous	Multihomogeneous Bézout number
	$\geq$
Polyhedral	BKK bound (mixed volume)

Can we beat the polyhedral homotopy?

Strategy & Results

The goal is to create a family of polynomial systems F_t such that:

(1) F_t has the same number of solutions for all $t\in\mathbb{C}$,
(2) $F_{t=1}=F$, and
(3) $F_{t=0}$ is a simple system to solve.

Our strategy is to use Anderson's *toric degeneration* from [1] for varieties which have an associated *finite Khovanskii basis*. The toric degeneration will guarantee the above three properties.

Khovanskii homotopy algorithm sketch

- Compute a finite Khovanskii basis associated to $\{F=0\}$ using the *SubalgebraBases* package for Macaulay2 [2, 3].
- Compute an embedding for Anderson's toric degeneration.
- Perform a weight degeneration on the resulting equations.
- Use resulting family of polynomials F_t as an optimal homotopy to compute $\{F=0\}$.

Khovanskii homotopy results

- Number of paths tracked is equal to the normalized volume of the *Newton-Okounkov body* associated to $\{F=0\}$, which can be less than the BKK bound.
- Homotopy is optimal for computing solutions to systems.

References

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[5] M. Plecnik. Flapping wing mechanism. Available at `https://www.youtube.com/watch?v=7aXmze9Ynis`.

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