Central Limit Theorems for Spatial Averages of Stochastic Heat Equation

Sefika Kuzgun

AWM Poster Session, Joint Mathematics Meetings 2022

Introduction

Consider the stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u\dot{W}, \qquad x \in \mathbb{R},\ t > 0 \tag{1}$$

where $u_0(x) = \delta_0(x)$

and $\dot{W}$ space-time white noise:

$$\mathrm{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta_0(t-s)\delta_0(x-y).$$

Existence of mild solution

There exists a unique measurable and adapted random field $u = \{u(t,x)\}_{(t,x)\in\mathbb{R}_+\times\mathbb{R}}$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[|u(t,x)|^p\right] < \infty\,,$$

and for all $t \geq 0$ and $x \in \mathbb{R}$

$$\begin{aligned} u(t,x) =& p_t(x) \\ &+ \int_{[0,t]\times\mathbb{R}} p_{t-s}(x-y)u(s,y)W(ds,dy). \end{aligned} \tag{2}$$

[Walsh '84, Chen & Dalang '15]

Renormalized solution

- The process $x \mapsto U(t,x) := u(t,x)/p_t(x)$ is stationary. [Amir, Corwin & Quastel '11]
- The equation (2) can be reformulated in terms of U: [Chen, Khoshnevisan, Nualart & Pu '20]

$$\begin{aligned} U(t,x) =& 1+ \\ &+ \int_{[0,t]\times\mathbb{R}} p_{\frac{\tau(t-\tau)}{t}}(\xi - \frac{\tau}{t}x)U(\tau,\xi)W(d\tau,d\xi). \end{aligned} \tag{3}$$

Spatial averages

- $\mathrm{E}\left[\int_{-R}^{R} U(t,x)dx\right] = 2R$
- $$\Sigma_{R,t}^2 := \mathrm{E}\left[\left(\int_{-R}^{R} U(t,x)dx - 2R\right)^2\right] \sim R\log R.$$
- Spatial averages:

$$G_{R,t} := \frac{1}{\Sigma_{R,t}}\left(\int_{-R}^{R} U(t,x)dx - 2R\right)$$

Malliavin calculus

- Integration by parts

$$\mathrm{E}\left[\langle DF, v\rangle_{\mathfrak{H}}\right] = \mathrm{E}\left[F\delta(v)\right]$$

- A crucial property: Any adapted and square integrable process v belongs to Dom δ and

$$\delta(v) = \int_{\mathbb{R}_+\times\mathbb{R}} v(s,y)W(ds,dy).$$

- In the aforementioned model, we have $G_{R,t} = \delta(v_{R,t})$ where

$$v_{R,t} = \frac{U(s,y)}{\Sigma_{R,t}}\int_{-R}^{R} p_{\frac{s(t-s)}{t}}(x - \frac{s}{t}y)dx$$

Estimates for the derivatives

- $$\|D_{s,y}U(t,x)\|_p \leq c_{T,p}\, p_{\frac{s(t-s)}{t}}(y - \frac{s}{t}x).$$
- $$\|D_{r,z}D_{s,y}U(t,x)\|_p \leq c_{T,p}\, p_{\frac{s(t-s)}{t}}(y - \frac{s}{t}x)\, p_{\frac{r(s-r)}{s}}(z - \frac{r}{s}y).$$
- $$\left\|\left(\langle DG_{R,t}, v_{R,t}\rangle\right)^{-1}\right\|_p \leq c_{t,p,\gamma}(\log R)^{\gamma}$$

Theorem

Let N denote a standard normal random variable. Then, for all $R \geq R_0$:

$$d_{\mathrm{TV}}(G_{R,t}, N) \leq \frac{C_t\sqrt{\log R}}{\sqrt{R}}.$$

[Chen, Khoshnevisan, Nualart & Pu '20]

Main Theorem

For all fixed $t > 0$ and $R \geq R_0$, $G_{R,t}$ has a density $f_{G_{R,t}}$. Moreover, let $\gamma > \frac{19}{2}$ and ϕ be the density of standard normal distribution, then, for all $R \geq R_0$:

$$\sup_{x\in\mathbb{R}} |f_{G_{R,t}}(x) - \phi(x)| \leq \frac{C_t(\log R)^{\gamma}}{\sqrt{R}}.$$

[K. & Nualart '21+]

Existence of density

- Assume that $F = \delta(v) \in \mathbb{D}^{2,6}$, $\mathrm{E}\left[F^2\right] = 1$ and

$$\mathrm{E}\left[\left|\langle DF, v\rangle_{\mathfrak{H}}\right|^{-4}\right] < \infty. \tag{4}$$

Then,

$$f_F(x) = \mathrm{E}\left[1\{F > x\}\,\delta\left(\frac{v}{\langle DF, v\rangle_{\mathfrak{H}}}\right)\right]$$

Malliavin-Stein method

- Let $F = \delta(v) \in \mathbb{D}^{1,2}$, N standard normal random variable.

$$d_{\mathrm{TV}}(F,N) \leq 2\sqrt{\mathrm{Var}\langle DF, v\rangle_{\mathfrak{H}}}.$$

- Let $F = \delta(v) \in \mathbb{D}^{2,6}$ satisfy (4), then

$$\begin{aligned} &\sup_{x\in\mathbb{R}} |f_F(x) - \phi(x)| \\ &\leq C\,\left\|\left(\langle DF, v\rangle_{\mathfrak{H}}\right)^{-1}\right\|_4^2 \left(\mathrm{E}\left[\left|\langle D\langle DF, v\rangle_{\mathfrak{H}}, v\rangle_{\mathfrak{H}}\right|^2\right]\right)^{1/2} \\ &+ \left(\|F\|_4 \left\|\left(\langle DF, v\rangle_{\mathfrak{H}}\right)^{-1}\right\|_4 + 2\right)\left(\mathrm{Var}\langle DF, v\rangle_{\mathfrak{H}}\right)^{1/2}. \end{aligned}$$

Other cases

- If $u_0 = 1$, and $\sigma(u)$ then

$$d_{TV}(F_{R,t}, N) \leq \frac{C_t}{\sqrt{R}},$$

$$\sup_{x\in\mathbb{R}} |f_{F_{R,t}}(x) - \phi(x)| \leq \frac{C_t}{\sqrt{R}}.$$

- If $u_0 = 1$, general dimension d,

$$\mathrm{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta_0(t-s)|x-y|^{-\beta}.$$

then

$$d_{TV}(F_{R,t}, N) \leq C_t R^{-\beta/2}$$

$$\sup_{x\in\mathbb{R}} |f_{F_{R,t}}(x) - \phi(x)| \leq C_t R^{-\beta/2}.$$

- Other equations like wave and fractional heat.

Additional Information

This is a joint work with David Nualart.

arXiv:2005.13676

Selected references

1. *An introduction to stochastic partial differential equations*, Walsh, John B, 1986
2. *Estimation of densities and applications*, Caballero, María Emilia and Fernández, Begoña and Nualart, David, 1998
3. *Stein's method on Wiener chaos*, Nourdin, Ivan and Peccati, Giovanni, 2008
4. *Probability distribution of the free energy of the continuum directed random polymer in 1+1 dimensions*, Amir, Gideon and Corwin, Ivan and Quastel, Jeremy, 2011
5. *On the chaotic character of the stochastic heat equation, before the onset of intermittency*, Conus, Daniel and Joseph, Mathew and Khoshnevisan, Davar, 2013
6. *Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions*, Chen, Le and Dalang, Robert C, 2015
7. *Regularity and strict positivity of densities for the nonlinear stochastic heat equation*, Chen, Le and Hu, Yaozhong and Nualart, David, 2016
8. *A central limit theorem for the stochastic heat equation*, Huang, Jingyu and Nualart, David and Viitasaari, Lauri, 2020