Hyperbolicity in Asymmetrical Lemon Billiards

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Abstract

Mathematical billiards are important models of dynamical systems from statistical mechanics. This project concerns a class of billiard tables called asymmetrical lemon billiards, which are convex tables formed by two circles of different radii. These billiards exhibit hyperbolicity despite violating the usual defocusing rule for billiards with concave boundaries. By varying two parameters, the radius of the larger circle and the distance between the centers of the circles, we classify different regimes which exhibit nonergodic behavior by proving the existence of elliptic islands. In addition, using a MATLAB simulation, the shape of the elliptic islands and strength of the Lyapunov exponents are analyzed numerically as a function of the parameters.

Asymmetrical Lemon Billiards

- Mathematical Billiard - a point particle moving at unit speed undergoing elastic reflections at collisions with a fixed boundary (assuming that the angle of incidence = angle of reflection)
- Lemon Billiard - a table created by the intersection of two circles of different radii, $r = 1$ and $R > 1$.
- d = the distance between the centers of the two circles.
- Must have $R - 1 < d < R + 1$ in order for the two circles to intersect.

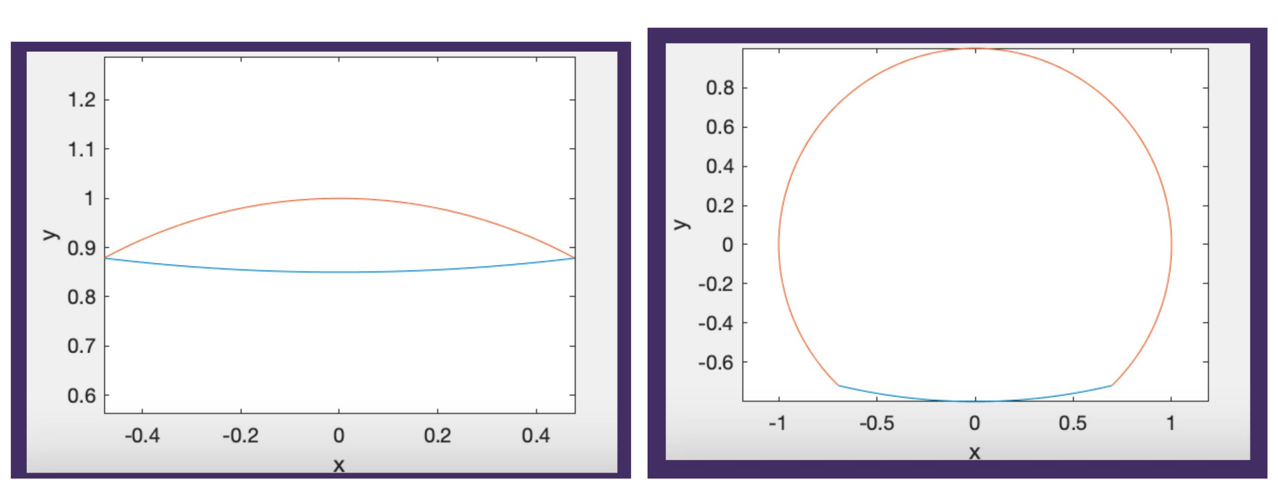

Figure: Lemon Billiard Tables

Motivating Fact: Lemon billiards can exhibit hyperbolic dynamics despite the apparent violation of the usual defocusing mechanism. (The distance between collisions with a concave boundary should be more than twice the radius of curvature.)

Research Questions

- For what values of R and d is the billiard ergodic?
- How does the central elliptic island vary as a function of R and d?
- How does the hyperbolicity of the table, expressed as the Lyapunov exponent, vary as a function of R and d?

Elliptic Islands

The presence of elliptic islands implies the table is not ergodic, so we began by determining for what values of R and d elliptic islands were present.

Period 2 Orbit. The main elliptic island in the lemon billiard is formed around the period 2 orbit through the center of the table. This orbit is hyperbolic for $d < R$, parabolic at $d = R$ and elliptic for $d > R$.

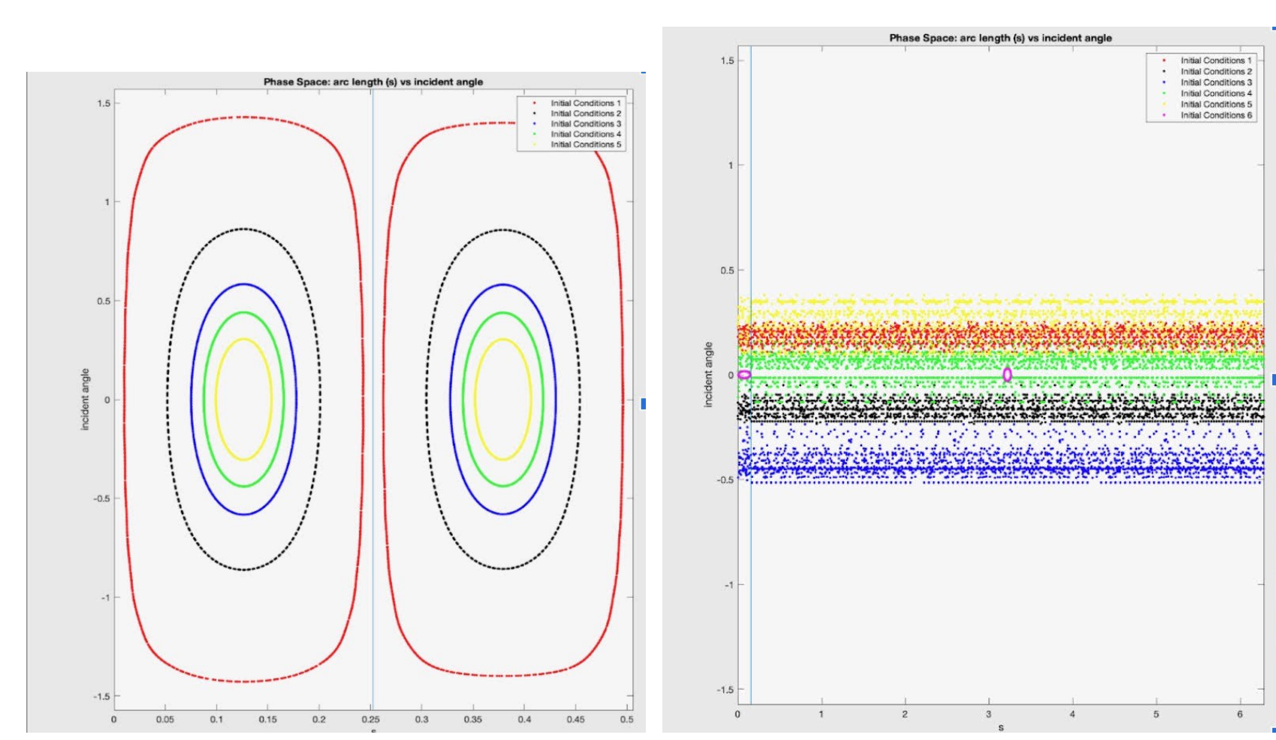

Figure: Lemon Billiard Elliptic Phase Spaces

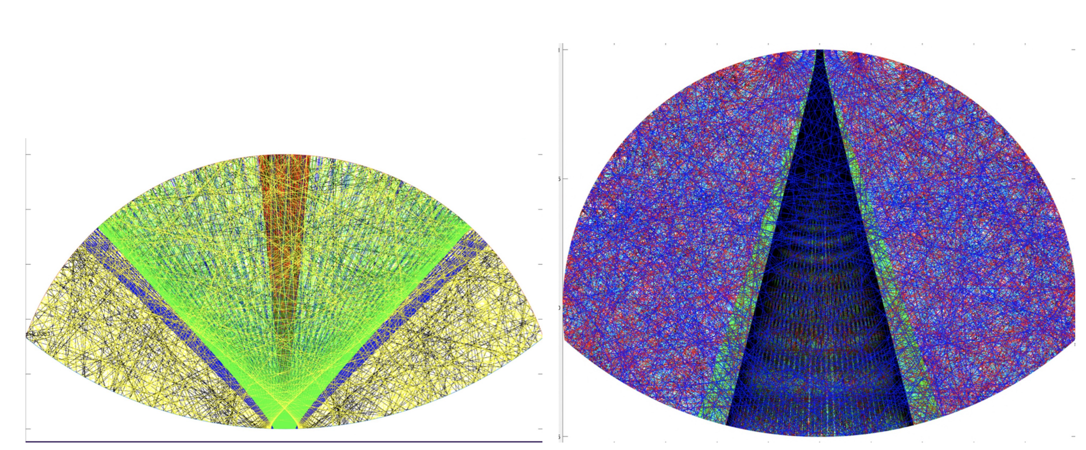

Figure: (Left)The Lemon Billiard Configuration space as d goes to R. (Right) The Lemon Billiard spaces as d goes to 1. Which are the transitions from Elliptic to Hyperbolic orbits, through a family of Parabolic orbits.

Period 3 Orbit. Exists and is elliptic in a parameter range even when the period 2 orbit is hyperbolic.

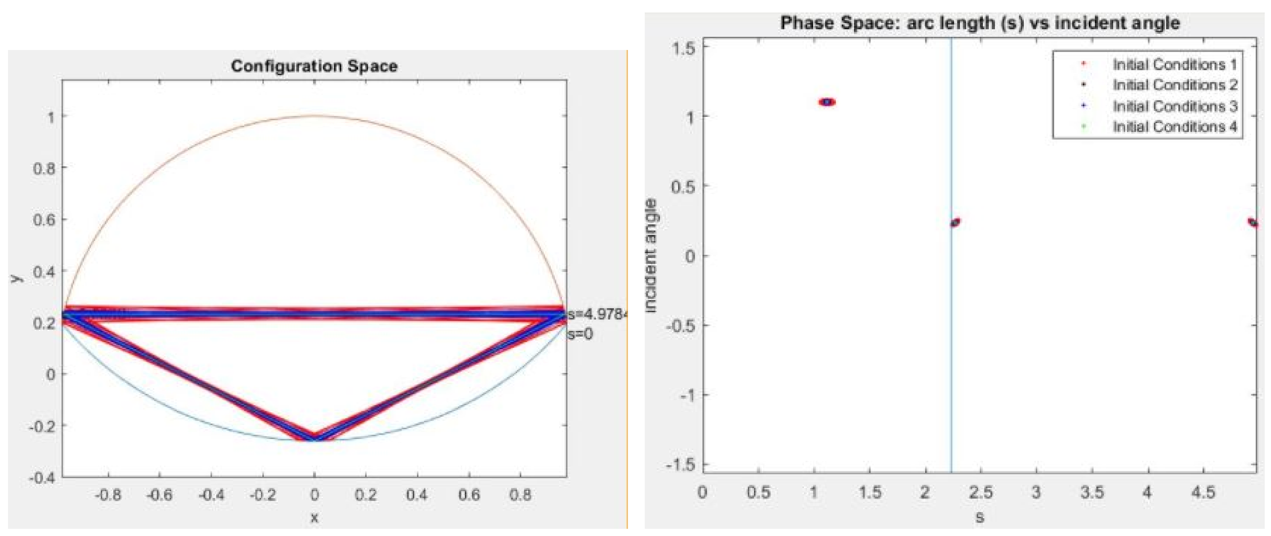

Figure: Period 3 Orbit

Conjecture

We conjecture that the boundary between ergodic and non-ergodic lemon billiards is when the boundary is when the circle of smaller radius is exactly a semi-circle in the table. This gives the equation $d = \sqrt{R^2 - 1}$. Then $d < \sqrt{R^2 - 1}$ should imply the table is ergodic, while $d > \sqrt{R^2 - 1}$ should imply the existence of elliptic islands.

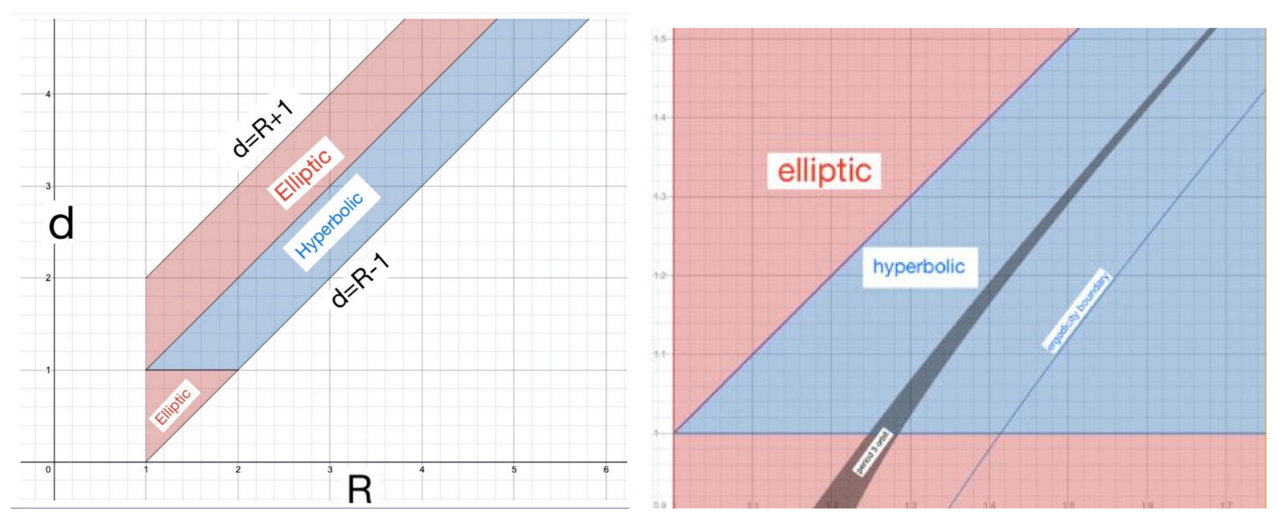

Figure: Lemon Billiard Parameter Space

Further Questions

- Finding other elliptic islands in the blue region of the period 2 orbit.
- Testing the conjecture for $d = \sqrt{R^2 - 1}$

Acknowledgements

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