Agreement proportion and boxicity for $(2,3)$-agreeable box societies

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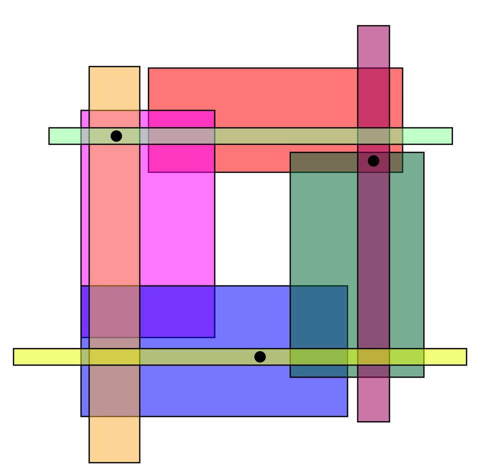
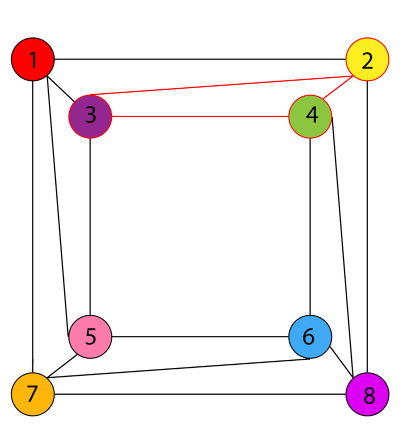
Following [3], in an agreeable society, members give ranges of acceptable opinions on d spectrum-based issues, so each member's "range of acceptance" can be represented by a d-dimensional box. This method of group decision-making using approval regions is called "approval voting." The properties of these boxes imply much about the society as results on approval voting can have real-world implications in the social sciences with applications ranging from voting models to online dating.

Members agree when their boxes overlap. A society S is **(q, p)-agreeable** if of any p members, q have agreement. Societies can also be represented by an **intersection graph** G whose vertices represent members and edges represent agreements. A society's **clique size** $\omega(G) = r$ is the size of the largest complete subgraph in its intersection graph. The **piercing number** $\tau(F)$ of a family of axis-parallel boxes corresponding to G is the fewest number of "piercings" for every box to be pierced, so the piercing number is equal to the **clique covering number** $\mathrm{cc}(G)$ of G, which is the fewest complete subgraphs needed to include every vertex. Here is an example of a $(2,3)$-agreeable 2-box society with $\tau(F) = 3$:



Boxicity and Agreement Proportion



Within the study of agreeable societies, much attention has been paid to lower bounds on boxicity and agreement proportion. A society's **boxicity** d is the lowest dimension in which it can be represented as a family of boxes. A society's **agreement proportion** α is $\omega(G)/|G|$.

Previously, the lowest known agreement proportion for a $(2,3)$-agreeable 2-box society was $3/8$ and which is shown above along with its intersection graph. However, the lower bound on agreement proportion was lower than $3/8$, so the bound's sharpness was unknown. Typically, **minimum agreement proportion** is represented ρ.

Lower Bound on Boxicity

In [2], the following theorem is proved for a lower bound on boxicity:

Theorem. Let G be a graph with no universal vertices and minimum degree δ. Then the boxicity of G has the lower bound:

$$\mathrm{box}(G) \geq \frac{n}{2(n-\delta-1)}$$

This bound was known not to be sharp as its fractional results imply a bound at the next greatest integer.

Lower Bound on Agreement Proportion

Using results on boxicity from [2], [1] proves the following lower bound on agreement proportion:

Theorem. For all $r \geq 1$ and $d \geq 1$, we have

$$\rho(r,d) \geq \frac{1}{2d}$$

For the society presented earlier, this result gives a lower bound of $1/4$ meaning that the lower bound may not be sharp.

Improved Lower Bounds

In [5], the following theorem connects agreeable societies much more closely with discrete geometry:

Theorem. For a box society S represented by a family of axis-parallel boxes F,

$$\rho(F) \geq \frac{1}{\tau(F)}$$

This theorem allows piercing numbers to be used to find lower bounds on agreement proportions. This is significant as piercing numbers have been studied more extensively than approval voting and have a concept analogous to (q,p)-agreeability: a family of boxes has the **(p, q)-property** if of any p boxes, q have a common point. A result on piercing numbers from [4] states:

Theorem. For a family F of axis-parallel d-boxes with the $(3,2)$-property, we have

$$\tau(F) \leq d+1$$

This is known to be sharp for $d \leq 5$. Synthesizing the results of [5] and [4], we improve the lower bound on agreement proportion as follows:

Theorem. For a $(2,3)$-agreeable d-box society,

$$\rho(d) \geq \frac{1}{d+1}$$

For the 2-box case, this provides an improved lower bound of $1/3$. The result from [4] can be modified to improve the lower bound on boxicity as well:

Theorem. For a $(2,3)$-agreeable box society with piercing number τ,

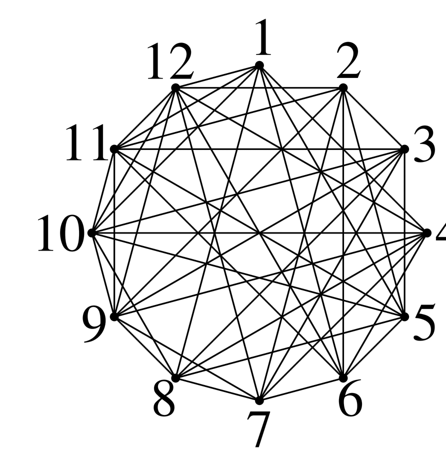
$$\mathrm{box}(G) \geq \tau - 1$$

Not only does this improve the bound found in [2], but this bound is also known to be sharp for $\tau \leq 6$. Because this lower bound is in terms of piercing number, it can be restated as:

$$\mathrm{box}(G) \geq \mathrm{cc}(G) - 1$$

This is of note because it allows for a lower bound on boxicity of a society to be calculated using only its intersection graph so long as its clique covering number or piercing number is less than or equal to 6.

The following graph has a clique size of 4 and 12 vertices meaning that it has an agreement proportion of $1/3$. Because the graph below is triangle-free, its complement must be $(2,3)$-agreeable. Additionally, it has a clique covering number of 3 because every vertex of the graph below can be included in one of three cliques: $\{1,4,10,12\}, \{2,5,6,11\}$, and $\{3,7,8,9\}$. This implies a boxicity of at least 2. In order to verify the boxicity, a family of boxes with this graph as its intersection graph must be constructed or an upper bound on boxicity equal to 2 must be found.



Future Research

Verifying the sharpness of the 2-box case currently stands as the most present task at hand. As this problem touches on graph theory, discrete geometry, and Ramsey theory, it presents many paths forward in pursuing more general lower bounds for agreement proportion and boxicity. Given previous precident for this problem, it is not unreasonable to think that finding a sharp lower bound on agreement proportion would be best accomplished by first finding a sharp lower bound on boxicity in terms of piercing number.

Acknowledgments

This project was supported by the Dr. M. Jean McKemie and Suzanne Mason Endowed Student/Faculty Fund for Innovative Mathematics Summer Scholarship.

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