Minimal Generating Sets of Determinantal Ideals in Alternating Matrices

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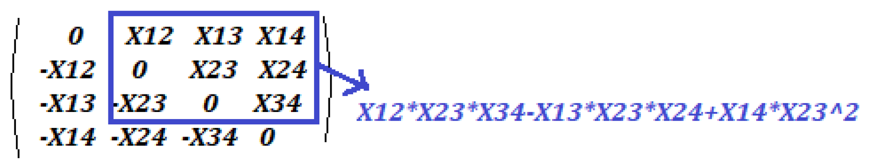
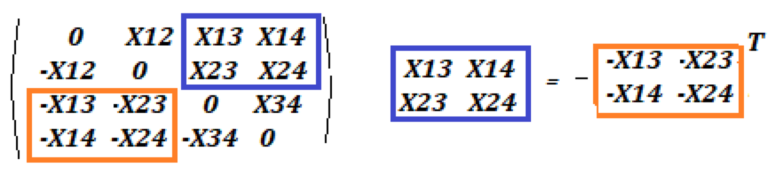
Introduction

Our project concerns the generic **alternating matrix**. This is a matrix X satisfying $X^T = -X$ where the entries above the matrix diagonal are distinct indeterminates, and the entries of the diagonal are all 0s. The generic alternating matrix A_n of size n is denoted as follows:

$$A_n = \begin{pmatrix} 0 & X_{12} & X_{13} & \cdots & X_{1n} \\ -X_{12} & 0 & X_{23} & \cdots & X_{2n} \\ -X_{13} & -X_{23} & 0 & \cdots & X_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -X_{1n} & -X_{2n} & -X_{3n} & \cdots & 0 \end{pmatrix}.$$

In the alternating matrix, we are specifically interested in the **minors**, which are the determinants of the submatrices. The ideal of t-minors of a matrix X is denoted as $I_t(X)$. We also denote the minor obtained from using rows $a_1, a_2, \ldots a_t$ and columns $b_1, b_2, \ldots, b_t$ as $[a_1, a_2, \ldots, a_t \mid b_1, b_2, \ldots, b_t]$. Below is an example of a 3×3 minor in the 4×4 alternating matrix.





Initial Results and Conjectures

The question that this project addresses is as follows:

Question

What is a minimal generating set for the determinantal ideal $I_t(A_n)$ for any $1 \leq t \leq n$?

Definition

*A minor $[a_1, a_2, \ldots, a_t \mid b_1, b_2, \ldots, b_t]$ is a **doset minor** if $a_i \leq b_i$ for all $1 \leq i \leq t$.*

Definition

***Initial-doset** t**-minors** are the minors $[a_1, a_2, \ldots, a_t \mid b_1, b_2, \ldots, b_t]$ such that for the first i such that $a_i \neq b_i$, we have that $a_i < b_i$, excluding principal minors if t is odd.*

We have shown that these minors generate our ideal.

Theorem (Hefty—Nguyen)

In an alternating matrix over a field not of characteristic 2, the initial-doset t-minors generate the ideal of t-minors.

Since we have a generating set, our next goal was to prove its minimality.

Conjecture

A minimal generating set for the t-minors of the $n \times n$ alternating matrix over a field of characteristic not equal to 2 is the set of initial-doset t-minors.

Theorem (Hefty—Nguyen)

The number of initial-doset t-minors of the $n \times n$ generic alternating matrix over fields with characteristic not equal to 2 is as follows: $\frac{\binom{n}{t}^2 - \binom{n}{t}}{2}$ when t is odd; $\frac{\binom{n}{t}^2 + \binom{n}{t}}{2}$ when t is even.

Characteristic 2 case: previous work by Aldo Conca gives that a minimal generating set for symmetric matrices is the doset minors. Since alternating matrices over fields of characteristic 2 are symmetric, we conjecture that

Conjecture

A minimal generating set for the t-minors of the $n \times n$ alternating matrix over a field of characteristic 2 is the doset t-minors, excluding principal minors if t is odd.

Unique Terms

A nice technique to show that our generating set is linearly independent would be to show that each minor in the set has a unique term (ignoring coefficients), since then no (vector space) linear combination of other minors could have that term. However, in the initial-doset 2-minors of the 4×4 alternating matrix, we have that $[1,3 \mid 2,4] = x_{12}x_{34} + x_{14}x_{23}$, $[1,2 \mid 3,4] = x_{13}x_{24} - x_{14}x_{23}$, and $[1,4 \mid 2,3] = -x_{12}x_{34} + x_{13}x_{24}$, meaning that we cannot choose a unique term for each of these three minors.

Algebras with Straightening Law

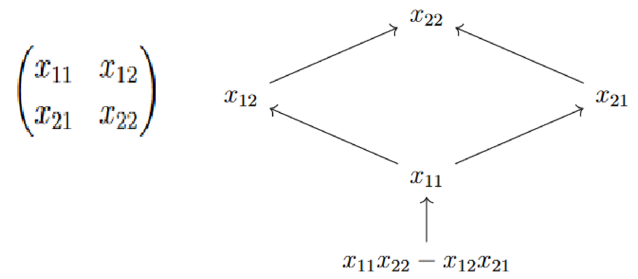
For the symmetric case, Aldo Conca used algebras with straightening law to show that the doset minors form a spanning set as a part of showing that they form a minimal generating set. The general idea of algebras with straightening law is as follows:

- Place a partial ordering on the minors
  - For example, for the symmetric case, we define a transitive relation $\preccurlyeq$ such that
$$[a_1, a_2, \ldots, a_t \mid b_1, b_2, \ldots, b_t] \preccurlyeq [c_1, c_2, \ldots, c_s \mid d_1, d_2, \ldots, d_s]$$
    if and only if $t \geq s$ and $b_i \leq c_i$ for all $1 \leq i \leq t$.
- Define standard monomials:

Definition

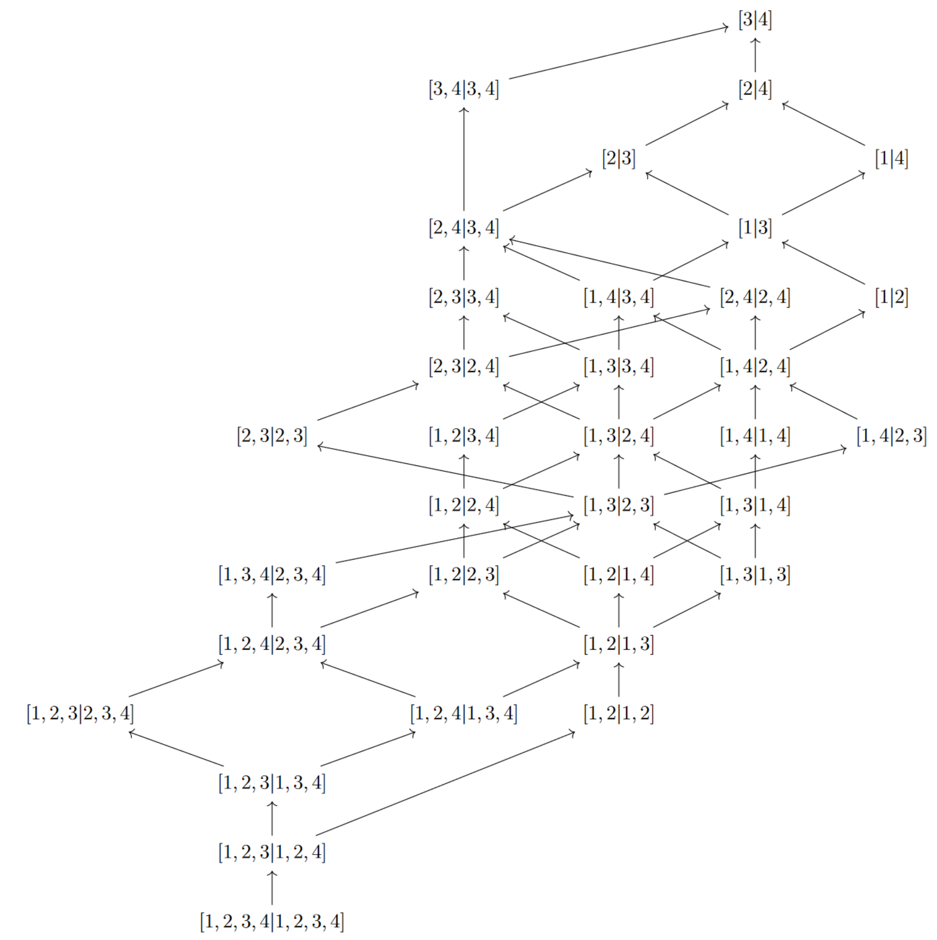
*A **standard monomial** is a product $M_1 M_2 \ldots M_n$ of minors where $M_1 \preccurlyeq M_2 \preccurlyeq \cdots \preccurlyeq M_n$.*

- The straightening law allows us to write a product of incomparable minors as a linear combination of standard monomials such that the smallest term in every one of those standard monomials is less than or equal to each of the term in the original product. Example:



$$[1|2] * [2|1] = x_{12}x_{21} = x_{11}x_{22} - (x_{11}x_{22} - x_{12}x_{21}) = [1|1] * [2|2] - [1,2|1,2].$$

We considered Conca's use of algebra with straightening law for our initial-doset minors. This strategy failed as the product of two incomparable minors $[1,4 \mid 2,3] \cdot [1,4 \mid 1,4]$ cannot be written as a linear combination of standard monomials such that each of the terms in the linear combination has a factor less than or equal to both $[1,4 \mid 2,3]$ and $[1,4 \mid 1,4]$. We can see this from the ordering diagram on the set of initial-doset minors of the 4×4 alternating matrix:



For our non-characteristic 2 case, since we failed to show that the initial-doset minors form a spanning set, we changed direction and focused on the characteristic 2 case. In the case of characteristic 2, alternating matrices are also symmetric matrices, so by Conca's result, the doset minors span the ideals as vector spaces. We then attempted to use KRS correspondence to prove linear independence.

KRS Correspondence

Through an algorithm using two functions **Delete** and **Insert** which are inverses of each other, Conca established a degree-preserving correspondence between the standard monomials (represented by d-tableaux) and normal monomials (represented by the 2-line array).

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 5 \\ \cline{1-2} \end{array} \longleftrightarrow \begin{pmatrix} 2 & 3 & 4 & 5 & 5 \\ 1 & 3 & 2 & 4 & 3 \end{pmatrix}$$

In this case, we have a correspondence between the standard monomial $[1,2,3 \mid 2,3,4] * [3,5 \mid 4,5]$ and the normal monomial $X_{12}X_{33}X_{24}X_{45}X_{35}$. However, this is where our problem lies. In our case, $X_{33} = 0$ because it is a variable on the diagonal. Thus, the algorithm sends a nonzero products of doset minors to 0, which is not an element of the basis of the normal monomials.

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References

[1] Bruns, Winfried and Vetter, Udo. *Determinantal rings*. Vol. 1327. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1988, pp. viii+236, 1988.

[2] Conca, Aldo. "Gröbner bases of ideals of minors of a symmetric matrix", In: *Journal of Algebra* 166.2 (1994), pp. 406-421.

[3] Grayson, Daniel R. and Stillman, Michael E. *Macaulay2, a software system for research in algebraic geometry*. Available at `http://www.math.uiuc.edu/Macaulay2/`